Wave Forcing of Submerged Elastic Plates

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1 Introduction

The problem of floating elastic plates has been well studied, while the problem of a submerged elastic plate has been much less well studied. We present here a solution for the submerged elastic plate based on the The Wiener-Hopf (WH) and Residue Calculus (RC) solution techniques. This work extends the solution presented at a previous workshop for a submerged porous plate [4, 5].

The Wiener-Hopf and Residue Calculus solution techniques have recently become widely used in the study of water-wave interactions with semi-infinite elastic plates [1, 10]. These studies, along with others using numerical methods like eigenfunction matching (EM) have been motivated by the usefulness of *floating* elastic plates in modelling scattering by sea-ice and man-made 'very large floating structures. Solutions for floating elastic plates of finite length have been obtained numerically using Green's functions [9, 6], but it has also been solved using the WH and RC methods [12].

In contrast, the problem of a submerged semi-infinite elastic plate subject to incident wave forcing has only been investigated once, by [7] using EM, although submerged rigid plates have received considerable attention, dating back to [2]. Recent work on submerged rigid plates is summarized in [8]. Other models of submerged plates that have been used are porous membranes and flexible membranes.

While the submerged elastic plate problem is somewhat idealized, it is likely that any practical structure which is sufficiently thin to be modelled as of negligible thickness could easily exhibit significant bending. The usefulness of submerged horizontal finite structures as breakwaters or wave barriers has received some research attention, including rigid plates (sometimes referred to as docks) and membranes and it has been suggested that it might also be useful to investigate an elastic plate rather than a membrane and this is the problem considered here. A major advantage of using horizontal plates (as opposed to structures with vertical extent) is the fact that they allow exchange of water and hardly disturb (horizontal) currents.

In the present work, which summarizes [11], we use WH to derive an analytic solution for the submerged semi-infinite elastic plate. We also show how the WH equations can be solved using RC. The steps can be reversed, so in this case the methods are equivalent, athough the same reasoning could also be applied to any WH/RC problem involving meromorphic dispersion relations. Results were checked numerically against [7], and against well-known limiting cases (rigid plate, floating elastic plate).

2 Problem formulation

The formulation of this problem has appeared in [7] and we only summarize here. Rectangular cartesian axes are chosen with the mean free-surface coinciding with the $(\bar{x}, \bar{y})$-plane and $\bar{z}$ measured vertically upwards. The fluid bottom is at $\bar{z} = -\bar{h}$. We assume invariance with respect to the $\bar{y}$-direction, so that the problem is two-dimensional (the extension to waves incident at an angle is not difficult, but is not discussed here). A submerged elastic plate of negligible thickness is placed along $\bar{z} = -\bar{d}$, $0 < \bar{x} < \infty$, $-\infty < \bar{y} < \infty$, where $-\bar{h} < -\bar{d} < 0$. We assume that all amplitudes are small enough that linear theory applies, and we make the usual assump-

tions that the fluid is inviscid, incompressible and irrotational. We denote the fluid velocity potential by $\overline{\phi}(\bar{x},\bar{y},\bar{z},t)$, and the displacement of the plate from its equilibrium postion by $\overline{W}(\bar{x},\bar{y},t)$. It is further assumed that all motion is time-harmonic with angular frequency ω, and that the motion is independent of the y-direction. Thus, we can write

$$\bar{\phi}(\bar{x},\bar{y},\bar{z},t) = \mathrm{Re}\left\{L^2\omega\phi(x,z)\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{1a}$$
$$\overline{W}(\bar{x},\bar{y},t) = \mathrm{Re}\left\{\mathrm{i}LW(x)\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{1b}$$

where Re denotes the real part, $\bar{x} = Lx$, $\bar{y} = Ly$, $\bar{z} = Lz$, and L is a natural length of the problem.

Under the assumptions above, $\phi(x,z)$ satisfies the Laplace equation in the fluid

$$\nabla^2\phi = \left(\partial_x^2 + \partial_z^2\right)\phi = 0, \tag{2}$$

with the boundary conditions

$$(\partial_z - \alpha)\,\phi(x,0) = 0 \quad \text{for } x \in \mathbb{R}, \tag{3a}$$
$$\partial_z\phi(x,-h) = 0 \quad \text{for } x \in \mathbb{R}, \tag{3b}$$
$$\partial_z\phi(x,-d) = W(x) \quad \text{for } x > 0, \tag{3c}$$
$$\left\langle\partial_z\phi(x,-d)\right\rangle = 0 \quad \text{for } x > 0, \tag{3d}$$
$$\Lambda(\partial_x)\partial_z\phi(x,-d) - \psi(x) = 0 \quad \text{for } x > 0, \tag{3e}$$

where $\Lambda(\partial_x) = \beta\lambda\partial_x^4 - \gamma$, $\psi(x) = \left\langle\phi(x,-d)\right\rangle$, and for a general function of z,

$$\left\langle\chi(z)\right\rangle = \chi(z^-) - \chi(z^+).$$

The non-dimensional parameters above are given by $\lambda = 1/\alpha$, $d = \bar{d}/L$, $h = \bar{h}/L$, and

$$\alpha = \frac{\omega^2 L}{g}, \quad \beta = \frac{EH^3}{12\rho g(1-\nu^2)L^4}, \quad \gamma = \frac{\rho_\mathrm{p} H}{\rho L}, \tag{4}$$

where g is the acceleration due to gravity, ρ_p is the density of the plate, D is the rigidity constant of the plate, and H is the thickness of the plate. β and γ are related to the stiffness and mass of the plate, respectively.

We also apply a plate-edge condition which implies that the pressure is continuous around the edge. This is given by $\psi(0) = 0$, or alternatively

$$\psi(x) \sim O\left(x^\upsilon\right), \tag{5}$$

where $x \sim 0$ and $\upsilon > 0$. We make no demands on υ at this stage although for a rigid or porous plate $\upsilon = \frac{1}{2}$ [5]. It is more difficult to prove what the value of υ should be when the thin-plate condition is applied, but by using an analytic solution method we are able to show that it takes the same value here also.

We also require two further edge conditions to complete the problem. If the plate is clamped, the edge conditions are

$$W(0) = \partial_x W(0) = 0, \tag{6}$$

and they read

$$\partial_x^2 W(0) = \partial_x^3 W(0) = 0. \tag{7}$$

for a plate whose edge is free to move. Finally we also need to apply a radiation condition, demanding that the scattered wave field consists of outgoing waves only.

3 Wiener-Hopf solution

We begin by using Green's theorem to derive and a Wiener-Hopf integral equation. Taking the Fourier transform, we then use the WH method to solve the resulting equation in Fourier space.

3.1 Green's theorem and solution of Wiener-Hopf equation

We begin with the well-known Green's function for open water of finite depth $G(x-x',z,z')$ which satisfies

$$\nabla^2 G(x-x',z,z') = \delta(x-x')\delta(z-z'), \tag{8a}$$
$$\partial_{z'}G(x-x',z,0) - \alpha G(x-x',z,0) = 0, \tag{8b}$$
$$\partial_{z'}G(x-x',z,-h) = 0. \tag{8c}$$

From [3], $\hat{G}$, the Fourier transform of G, is given by

$$\hat{G}(k,z,z') = \int_{-\infty}^{\infty} G(x,z,z')\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = \frac{\varphi_0(z_+,k)}{f_\mathrm{ow}(k)}\varphi_h(z_-,k), \tag{9}$$

where $z_+ = \max\{z,z'\}$, $z_- = \min\{z,z'\}$,

$$\varphi_0(z,k) = \lambda\cosh(kz) + \frac{1}{k}\sinh(kz), \tag{10a}$$
$$\varphi_h(z,k) = \cosh(k(z+h)), \tag{10b}$$

$\lambda = 1/\alpha$, and $f_\mathrm{ow}(k) = \cosh(kh) - \lambda k\sinh(kh)$. f_ow has zeros at $k = \pm k_n$ where the k_n are defined so that $\mathrm{Arg}[k_n] \in [0,\pi)$.

We now use Green's theorem to find an integral equation in $\psi(x)$. Initially we find that

$$\phi^S(x,z) = \int_0^\infty \partial_{z'} G(x-x',z,-d)\psi(x')\mathrm{d}x', \tag{11}$$

where $\phi^S = \phi - \phi^I$, ϕ^I represents the potential due to the incident wave, and is given by $\mathrm{e}^{\mathrm{i}k_0 x}\phi_0(z)$, where $\phi_n(z) = \cosh(k_n(z+h))$. If we now apply $\Lambda(\partial_x)\partial_z = \big(\beta\lambda\partial_x^4 - \gamma\big)\partial_z$ to (11), let $z \to -d$, and apply (3e) we find

$$\psi(x) = A\mathrm{e}^{\mathrm{i}k_0 x} + \int_0^\infty K(x-x')\psi(x')\mathrm{d}x', \tag{12}$$

where $A = \Lambda(k_0)\phi_0'(-d)$, $x > 0$ and

$$K(x) = \Lambda(\partial_x)\partial_z\partial_{z'}G(x-x',-d,-d).$$

Extending the equation to $x < 0$ in the usual way and taking the Fourier transform then gives

$$\frac{f_{\mathrm{sp}}(k)}{f_{\mathrm{ow}}(k)}\Psi^+(k) + \Psi^-(k) = \frac{\mathrm{i}A}{k+k_0}, \tag{13}$$

where

$$f_{\mathrm{sp}}(k) = f_{\mathrm{ow}}(k) - \Lambda(k)C(k), \tag{14a}$$
$$C(k) = \varphi_0'(-d,k)\varphi_h'(-d,k), \tag{14b}$$
$$\Psi^+(k) = \int_0^\infty \psi(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x, \tag{14c}$$
$$\Psi^-(k) = \int_{-\infty}^0 \Lambda(\partial_x)\partial_z\phi^S(x,-d)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x. \tag{14d}$$

$\Psi^\pm$ are unknown functions that are analytic in the regions $\mathbb{C}^\pm$, where $\mathbb{C}^+ = \big\{k \in \mathbb{C}\,\big|\,0 \le \mathrm{Arg}[k] < \pi\big\}$, and $\mathbb{C}^- = \mathbb{C} - \mathbb{C}^+$.

Before we commence to solve (13), we note that to compute ϕ we do not actually have to find an explicit expression for ψ, but rather only need to find Ψ^+ (indeed we do not even need to find Ψ^-). First we note that from (11)

$$\begin{aligned}\Psi(k,z) &= \int_{-\infty}^\infty \phi^S(x,z)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x \\ &= \partial_{z'}\hat{G}(k,z,-d)\Psi^+(k).\end{aligned} \tag{15}$$

Thus, having first found Ψ^+, inverting the transform Ψ will enable us to find an expression for the potential everywhere in the fluid domain.

Now, the dispersion relation for the submerged plate is $f_{\mathrm{sp}}(k) = 0$. It has two positive real roots $\kappa_0 > \kappa_{-1}$, two roots κ_{-2} and $\kappa_{-3} = -\kappa_{-2}^*$ which are usually complex (but can become imaginary), infinitely many imaginary roots κ_n $(n = 1, 2, \ldots)$, and the negatives of these roots. As with the floating elastic plate, there can be double and triple roots on the imaginary axis, but these cases are fairly rare, so we do not allow for them.

To solve (13), we factorize $f_{\mathrm{sp}}/f_{\mathrm{ow}} = K^+(k)/K^-(k)$, where $K^\pm$ are functions that are analytic in $\mathbb{C}^\pm$ respectively. We compute them as

$$K^+(k) = \frac{\mathrm{e}^{\chi(k)}\prod_{n=-3}^\infty (1+k/\kappa_n)\,\mathrm{e}^{-k/\kappa_n}}{\prod_{n=0}^\infty (1+k/k_n)\,\mathrm{e}^{-k/k_n}}, \tag{16}$$

and $K^-(k) = 1/K^+(-k)$, where

$$\chi(k) = (\mathrm{i}k/\pi)\big(h\log(h) - c\log(c) - d\log(d)\big).$$

These infinite products can be made to converge rapidly, and they are $O(k^{\pm 5/2})$ as $|k| \to \infty$.

After the factorisation has been completed, (13) becomes

$$\begin{aligned}K^+(k)\Psi^+(k) &= \mathrm{i}A\frac{K^-(k)}{k+k_0} - K^-(k)\Psi^-(k) \\ &= Q(k) + \mathrm{i}A\frac{K^-(-k_0)}{k+k_0},\end{aligned} \tag{17}$$

where $Q(k)$ is an entire function (by analytic continuation). From (5), as $k \to \infty$, $K^+(k)\Psi^+(k) \sim O\big(k^{3/2-\upsilon}\big)$, so by Liouville's theorem $\upsilon = \frac{1}{2}$ and $Q(k) = \gamma_1 + \gamma_2 k$ is an unknown linear function. Thus, having found Ψ^+, we can invert (15) to find ϕ in terms of the γ_j; these are found by applying either the clamped (6) or free (7) edge conditions. ϕ can eventually be written as

$$\phi^S(x,z) = \sum_{n=0}^\infty a_n \mathrm{e}^{-\mathrm{i}k_n x}\phi_n(z) \quad \text{for } x < 0, \tag{18a}$$
$$\phi(x,z) = \sum_{n=-3}^\infty b_n \mathrm{e}^{-\mathrm{i}\kappa_n x}\psi_n(z) \quad \text{for } x > 0, \tag{18b}$$

where $\psi_n(z) = F(z,\kappa_n)$, and

$$F(z,k) = \begin{cases}\varphi_0(z,k)\partial_z\varphi_h(-d,k) & \text{for } z > -d, \\ \varphi_h(z,k)\partial_z\varphi_0(-d,k) & \text{for } z < -d.\end{cases}$$

4 Derivation of Residue Calculus equations

In this section we derive the RC equations for this problem from the WH equation (13). Due to lack of space, we do not solve them here but details can be found in [11], which shows that doing so gives the same solution as the WH method. Note that the steps below are reversible, so in this case the methods are equivalent. (Also note that the same reasoning could be applied to any problem involving meromorphic dispersion relations.)

Before we begin, let us define for future reference the following singular functions:

$$S_0(k) = \sum_{n=0}^{\infty} \left(\frac{f_{\rm sp}(k_n)\Psi^+(-k_n)}{f'_{\rm ow}(k_n)(k+k_n)} - \frac{f_{\rm sp}(k_n)\Psi^+(k_n)}{f'_{\rm ow}(k_n)(k-k_n)} \right),$$

$$S^-(k) = -\mathrm{i}\sum_{n=0}^{\infty} \frac{\hat{a}_n}{k-k_n}, \quad S^+(k) = \mathrm{i}\sum_{n=-3}^{\infty} \frac{b_n f_{\rm ow}(\kappa_n)}{k+\kappa_n},$$

where $\hat{a}_n = a_n \Lambda(k_n)\phi'_n(-d)$, and the residues in S_1 and S_2 are written so that a_n and b_n correspond to (18). Now, (13) implies that $\Psi^\pm - S^\pm$ are analytic, and since $\Psi^+ \to 0$ as $|k| \to \infty$, $\Psi^+ = S^+$. We can rewrite (13) as

$$\frac{A}{k+k_0} + S_1(k) - \Psi^-(k) + S_0(k) - S_1(k) = \frac{f_{\rm sp}(k)}{f_{\rm ow}(k)}\Psi^+(k) + S_0(k), \quad (19)$$

where the right hand side is now analytic, which implies the left hand side is also. Making the residue at each pole on the left hand side zero we obtain two sets of equations:

$$A\delta_{n0} = -\frac{f_{\rm sp}(k_n)\Psi^+(-k_n)}{f'_{\rm ow}(k_n)}, \quad (20a)$$

$$\hat{a}_n = -\frac{f_{\rm sp}(k_n)\Psi^+(-k_n)}{f'_{\rm ow}(k_n)}. \quad (20b)$$

(20a) is an infinite system of linear equations in the b_n coefficients which may be solved with residue calculus, or numerically by truncation (which provides a useful test, although only very slow convergence can be obtained in this way). [11] also shows that these equations are the same as those obtained by EM.

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