Two-dimensional deterministic and stochastic evolution equations for shoaling of nonlinear waves

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Introduction Nonlinear interactions between sea waves and the bottom are a main mechanism of energy transfer between the different wave frequencies in the near-shore region. In this region, nonlinear interactions act much faster than in deep water due to quadratic resonance interactions. One of the methods for solving this flow regime is using quadratic nonlinear mild-slope (MS) type wave models. These models consist of a linear mild-slope type equation for each wave harmonic coupled by quadratic nonlinear terms to all other harmonics. The derivation of these models assumes in a heuristic manner that the wave's phase function is an integral of its wavenumber (see e.g. [1, 5, 10]). In the present work, a general phase function is applied and selected according to the required resulting model type. We chose two basic types of phase function definitions, a linear one and a nonlinear one, which create two basic types of models. This results in improved two-dimensional extensions of the model of [1]. In addition, these models retain higher order terms, which were formerly neglected. In order to reduce computational costs, a simplification of the two-dimensional evolution models is presented in accordance with the dominating resonance mechanism—the class III Bragg resonance.

The wavenumber vectors and the phase functions are needed for the construction of this type of models. This limits these models to one-dimensional ones, or to two dimensional ones with some crude assumptions. In the present work, governing equations for the wavenumber vectors and the phase functions are constructed in order to allow solving them together with the evolution equations using an iterative method. Furthermore, in order to simplify the solution procedure, their perturbation solution is derived for the case of oblique incident waves with mild lateral bottom changes. The perturbation solution enables creating evolution models that include lateral bottom changes in the nonlinear phase mismatch. This addition, which was formerly neglected or averaged out in both deterministic and stochastic MS-type models, should greatly improve the modeling of two-dimensional nonlinear shoaling problems. It is applied here not only to the new deterministic models but also for improving the deterministic parabolic model of [5] and as a basis for two-dimensional stochastic models. Finally, more accurate stochastic (phased-averaged) two-equation and one-equation wave evolution models (i.e. with and without an explicit equation for the bi-spectra) are constructed for oblique incident waves, which undergo two-dimensional nonlinear triad interactions.

Linear and nonlinear eikonal equation models The irrotational flow of an incompressible fluid with a free surface can be described by the Laplace equation of the velocity potential together with boundary conditions in the following manner

$$\nabla^2\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0, \qquad -h \le z \le 0, \tag{1}$$

$$\frac{\partial\Phi}{\partial t} + g\eta + \frac{1}{2}|\nabla\Phi|^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2 + \eta\frac{\partial^2\Phi}{\partial z\partial t} = O\left(\varepsilon^3\right), \qquad z = 0, \tag{2}$$

$$\frac{\partial\eta}{\partial t} - \frac{\partial\Phi}{\partial z} + \nabla\Phi\cdot\nabla\eta - \eta\frac{\partial^2\Phi}{\partial z^2} = O\left(\varepsilon^3\right), \qquad z = 0, \tag{3}$$

$$\frac{\partial\Phi}{\partial z} + \nabla\Phi\cdot\nabla\eta = 0, \qquad z = -h, \tag{4}$$

where $\Phi(x,y,z,t)$ is the wave velocity potential; $\eta(x,y,t)$ is the free-surface displacement; $h(x,y)$ is the bottom profile, and ∇ is the horizontal gradient operator. The vertical coordinate z of the Cartesian coordinate system $Oxyz$ directs upward with the Oxy-plane located on the still-water surface. The dynamic and kinematic boundary conditions given in equations (2) and (3) were approximated to be written on the still water level up to quadratic order in wave steepness ε, as was done by [1, 5].

Let us assume a two-dimensional nearly time-harmonic wave propagation with a pre-defined vertical profile

$$\Phi(x,y,z,t) = \sum_{n=1}^{N} f_n(k_n,h,z)\left(\phi_n(x,y,t)e^{-\mathrm{i}\omega_n t} + \phi_n^*(x,y,t)e^{\mathrm{i}\omega_n t}\right), \qquad f_n = \frac{\cosh k_n(z+h)}{\cosh k_n h}. \tag{5}$$

Here, $*$ denotes the complex conjugate; i denotes the imaginary unit number, and k_n is calculated using the linear dispersion relation $\omega_n^2 = gk_n \tanh k_n h$. In addition, the complex potential base functions can be written in a traveling wave form as

$$\phi_n(x,y,t) = B_n(x,y,t)e^{\mathrm{i}S_n(x,y)}. \tag{6}$$

Applying equations (1)-(6) to Green's second identity relating Φ and f_n in the manner of [9, 5] while separating the result to its harmonics allows writing a set of nonlinear evolution equations as follows

$$\begin{aligned}&\nabla\cdot\left[(CC_g)_n\nabla B_n\right] + \left[\left(k_n^2 - |\nabla S_n|^2\right)(CC_g)_n + gR_n\right]B_n\\ &\quad -\frac{\partial^2 B_n}{\partial t^2} + 2\mathrm{i}\omega_n\frac{\partial B_n}{\partial t} + \mathrm{i}\nabla\cdot\left[(CC_g)_n B_n\nabla S_n\right] + \mathrm{i}(CC_g)_n\nabla B_n\cdot\nabla S_n = \mathrm{NL}_n.\end{aligned} \tag{7}$$

Here, NL_n represents the nonlinear part of the equation related toh armonic ω_n. It is given as

$$\begin{aligned}\mathrm{NL}_n \;=\; & -\frac{\mathrm{i}}{4}\sum_{l=1}^{n-1}\Big[-2\omega_n\nabla S_l\cdot\nabla S_{n-l} + \mathrm{i}\omega_l\nabla^2 S_{n-l} - \omega_l|\nabla S_{n-l}|^2 + \mathrm{i}\omega_{n-l}\nabla^2 S_l - \omega_{n-l}|\nabla S_l|^2\\ &+\frac{\omega_l\omega_{n-l}\omega_n}{g^2}\left(\omega_l^2 + \omega_l\omega_{n-l} + \omega_{n-l}^2\right)\Big]B_l B_{n-l}e^{\mathrm{i}(S_l+S_{n-l}-S_n)}\\ &-\frac{\mathrm{i}}{2}\sum_{l=1}^{N-n}\Big[2\omega_n\nabla S_l\cdot\nabla S_{n+l} - \mathrm{i}\omega_{n+l}\nabla^2 S_l + \omega_l|\nabla S_{n+l}|^2 - \mathrm{i}\omega_l\nabla^2 S_{n+l} - \omega_{n+l}|\nabla S_l|^2\\ &-\frac{\omega_l\omega_{n+l}\omega_n}{g^2}\left(\omega_l^2 - \omega_l\omega_{n+l} + \omega_{n+l}^2\right)\Big]B_l^* B_{n+l}e^{\mathrm{i}(S_{n-l}-S_l-S_n)}.\end{aligned} \tag{8}$$

The following eikonal equations can be formulated from the real part of equation (8):

$$|\mathbf{K}_n|^2 \;\equiv\; |\nabla S_n|^2 = k_n^2 + \frac{1}{(CC_g)_n B_n}\left\{-\frac{\partial^2 B_n}{\partial t^2} + \nabla\cdot\left[(CC_g)_n\nabla B_n\right]\right\} + \frac{gR_n}{(CC_g)_n}, \tag{9}$$

$$|\mathbf{K}_n|^2 \;\equiv\; |\nabla S_n|^2 = k_n^2 + \frac{1}{(CC_g)_n B_n}\left\{-\frac{\partial^2 B_n}{\partial t^2} + \nabla\cdot\left[(CC_g)_n\nabla B_n\right]\right\} + \frac{gR_n}{(CC_g)_n} - \Re[\mathrm{NL}_n], \tag{10}$$

where $R_n = r_{1,n}\nabla^2 h + r_{2,n}|\nabla h|^2$ is the higher order bottom derivatives component given by [2], and $\Re$ denotes the real part. In former works of [1, 3, 6, 10] the definition (9) was taken as $|\nabla S_n|^2 = k_n^2$ without a formal derivation of the eikonal equation while neglecting all other terms in equation (9) as well as neglecting the real terms in equation (8). For a linear time-harmonic model, it was shown that using the definition (9) even without its last term yields a significant increase in accuracy (see e.g. [8]).

For the derivation of nonlinear evolution equations, the phase function can be chosen for the linear part as in equation (9) or (10). For the nonlinear part, equation (9) can be approximated to hold for the leading order resonance—the class III Bragg resonance (see [10]), where the following lower order effective wave number is sufficient

$$\left|\mathbf{K}_n^{\mathrm{L}}\right|^2 = k_n^2 + \frac{gr_{1,n}\nabla^2 h}{(CC_g)_n}. \tag{11}$$

The transport equations for the two types of eikonal equations (9) and (10) are given as

$$2\omega_n\frac{\partial B_n}{\partial t} + 2(CC_g)_n\nabla B_n\cdot\mathbf{K}_n + \nabla\cdot\left[(CC_g)_n\mathbf{K}_n\right]B_n \;=\; \mathrm{NL}_n, \tag{12}$$

$$2\omega_n\frac{\partial B_n}{\partial t} + 2(CC_g)_n\nabla B_n\cdot\mathbf{K}_n + \nabla\cdot\left[(CC_g)_n\mathbf{K}_n\right]B_n \;=\; \Im[\mathrm{NL}_n], \tag{13}$$

where $\Im$ denotes the imaginary part. In order to solve these transport equations, some knowledge is missing—the direction of the vector $\mathbf{K}_n$ and the definition of S_n. Former works assumed $S_n = \int k_n(h(x))\,dx$, which applied for a one-dimensional propagation [1, 6] or a two dimensional one using an averaged bottom profile [5].

Let us inspect the wavenumber and phase fields for waves propagating over a general bottom. From the definition of $\mathbf{K}_n$ it can be seen that

$$0 \equiv \nabla\times\nabla S_n = \nabla\times\mathbf{K}_n = \frac{\partial K_{1,n}}{\partial y} - \frac{\partial K_{2,n}}{\partial x}, \qquad \mathbf{K}_n = (K_{1,n}, K_{2,n})^T, \tag{14}$$

where for forward propagating waves

$$K_{1,n} = \sqrt{K_n^2 - K_{2,n}^2}. \tag{15}$$

Equation (15) can be substituted to equation (14) in order to yield a single nonlinear PDE for K_2. The corresponding phase function relates to K_1 in the following manner

$$S_x = K_1, \qquad S = \int_{x_0}^{x} K_1(\xi_1, y)\,d\xi_1 + \gamma(y). \tag{16}$$

Equation (16) can be used together with the boundary conditions to find the constants of integration. A wave maker boundary condition at x_0 yields $\gamma(y) = S(x_0, y)$, where $S(x_0, y)$ is the known boundary phase of the generated wave.

A perturbation solution for the wavenumber vectors and the phase functions for oblique incident waves with mild lateral bottom changes The aim of this section is to derive analytical solutions for the wavenumber vectors and phase functions for the case of mild lateral bottom changes. Equations (14) and (15) govern the wave number vector field for forward propagating waves (in our case, the positive x-direction), but unfortunately they consists of a partial differential equation and a nonlinear algebraic relation. Taking into account mild lateral bottom changes, the scaling of the wavenumber components can be taken as follows

$$K_n = K_n(x, \delta y, t), \quad K_{1,n} = K_{1,n}(x, \delta y, t), \quad K_{2,n} = K_{2,n}(\delta x, \delta y, t), \quad K_{2,n} - K_{02,n} = O(\delta), \quad \delta \ll 1. \tag{17}$$

Applying Taylor series to equation (15) around $K_{02,n}$ (the effective wavenumber value on the lateral direction in deep water) and substituting it to equation (14) allows writing a differential equation for K_2:

$$\frac{K_n}{\sqrt{K_n^2 - K_{02,n}^2}} \frac{\partial K_n}{\partial y} \delta - \frac{\partial K_{2,n}}{\partial x} \delta + \frac{K_{02,n}}{\left(K_n^2 - K_{02,n}^2\right)^{3/2}} \left[K_n (K_{2,n} - K_{02,n}) \frac{\partial K_n}{\partial y} + \left(K_n^2 - K_{02,n}^2\right) \frac{\partial K_{2,n}}{\partial y} \right] \delta^2 = O\left(\delta^3\right). \tag{18}$$

Equation (18) is still a partial differential equation for K_2, but as its derivatives for each direction are in different orders, they can be easily separated to two hierarchic ordinary differential equations. In order to do so, a perturbation expansion is applied in the following manner

$$K_{2,n} = \beta_{0,n} + \delta \beta_{1,n} + \delta^2 \beta_{2,n} + \ldots \tag{19}$$

Substituting equation (19) to equation (18) and solving for each order leads to

$$\begin{aligned} K_{2,n} \;=\; & K_{02,n} + \int_{x_0}^{x} \frac{K_n}{\sqrt{K_n^2 - K_{02,n}^2}} \frac{\partial K_n}{\partial y} d\xi + \int_{x_0}^{x} d\xi \frac{K_{02,n}}{2\left(K_n^2 - K_{02,n}^2\right)^{3/2}} \\ & \times \left[K_n (K_{2,n} - K_{02,n}) \frac{\partial K_n}{\partial y} + \left(K_n^2 - K_{02,n}^2\right) \frac{\partial}{\partial y} \int_{x_0}^{x} \frac{K_n}{\sqrt{K_n^2 - K_{02,n}^2}} \frac{\partial K_n}{\partial y} d\xi \right] + O\left(\delta^2\right). \end{aligned} \tag{20}$$

The solution for K_1 can now be found by substituting equation (20) to the Taylor expansion of equation (15):

$$K_{1,n} = \sqrt{K_n^2 - K_{02,n}^2} - \frac{K_{02,n}}{\sqrt{K_n^2 - K_{02,n}^2}} \int_{x_0}^{x} \frac{K_n}{\sqrt{K_n^2 - K_{02,n}^2}} \frac{\partial K_n}{\partial y} d\xi + O\left(\delta^2\right). \tag{21}$$

This results in solving the entire wavenumber field. Still, for a nonlinear model, the phase function is also needed. The phase function S is the integral of K_1 and K_2 in the x and y directions respectively, hence it is given by using equation (16) as

$$S \;=\; K_{02,n} y + \int_{x_0}^{x} \sqrt{K_n^2 - K_{02,n}^2} d\xi - \int_{x_0}^{x} \frac{K_{02,n}}{\sqrt{K_n^2 - K_{02,n}^2}} \int_{x_0}^{\xi_1} \frac{K_n}{\sqrt{K_n^2 - K_{02,n}^2}} \frac{\partial K_n}{\partial y} d\xi d\xi_1 + \gamma_n + O\left(\delta^2\right). \tag{22}$$

Let us further inspect equations (20)-(22). Assuming no changes in the lateral direction y, degenerate these equations to the case of oblique incident waves on a beach with straight depth contours parallel to the shore line. For the case of mild changes in the lateral direction, even a lower order of the perturbation solution (i.e. up to β_0) gives an extension. This simple case can be written as

$$K_{1,n} = \sqrt{K_n^2 - K_{02,n}^2}, \quad K_{2,n} = K_{02,n} + \int_{x_0}^{x} \frac{K_n}{\sqrt{K_n^2 - K_{02,n}^2}} \frac{\partial K_n}{\partial y} d\xi, \quad S_n = K_{02,n} y + \int_{x_0}^{x} \sqrt{K_n^2 - K_{02,n}^2} d\xi + \gamma_n, \tag{23}$$

and the extension is apparent as a non-constant term in the definition of $K_{2,n}$ and in the dependency of S_n and

K_n on the lateral direction y.

In other related works, when there are lateral bottom changes, some further simplifications are applied such as assuming k as a constant in the lateral direction by taking a y-averaged bottom profile [5, 10] and assuming small crossing angles [4]. In addition to an extension to oblique incident waves, equations (23) clearly show that it is better not to average the profile, and there is no need of any assumptions regarding the crossing angles of different wave components.

Note that equation (18) can be easily extended to allow calculating also higher order β-terms. Another option for increasing the accuracy of the solution is to use a Padé approximation instead of a Taylor one. This allows creating a more accurate equation while using lower order terms.

Two-dimensional nonlinear evolution models Equations (20)-(22) give the information needed in order to formulate the nonlinear transport equations under the assumption of a bottom with mild changes in the lateral direction. When these equations are taken together with equations (9) and (12), they yield complex evolution equations. Whereas, when they are taken together with equations (10) and (13), they yield real evolution equations.

In order to allow for an a-priori solution of the wavenumber vector and the phase function, we can choose a simple eikonal equation that does not depend on the amplitudes thus making the numerical computation easier, as no iterative method is required. As an example, let us assume

$$\mathbf{K}_n^2 = k_n^2 + \frac{gR_n}{(CC_g)_n}, \tag{24}$$

which results by neglecting the second term in equation (9). The wave number vector and the phase functions are presented in equations (20)-(22), where the transport equation (12) and the simplified eikonal equation (24) can be solved as the evolution equations for the flow.

For more accurate but more complicated models, where the eikonal equation cannot be solved a-priori, an iterative method can be used such as the one of [7]. The evolution equations can be written in terms of the amplitudes of the surface elevation instead of the ones of the velocity potential in the same manner of [3]. Numerical example of the deterministic models and the relating stochastic models will be presented in the lecture.

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