THE INFLUENCE OF THE ICE COVER ON THE UNIFORM MOTION OF A SUBMERGED SPHERE

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1. Introduction

In recent years the increase of human activities in the polar regions amplified the necessity of investigations in the domain of ice cover dynamics. In particular, the problem of response of the floating ice on the loading induced by a moving submerged body became acute. Considerable study has been given to a behavior of the ice cover's three-dimensional flexural oscillations caused by a moving pressure area (see, for example, Squire *et al.*, 1996 and Bukatov & Zharkov, 1997). However, to the author's knowledge, the effect of the submerged body on the ice cover is still not fully understood.

This paper studies wave patterns generated by a steadily moving submerged sphere in the framework of the linearized theory. Ice and water are assumed to be homogeneous and of infinite horizontal extent An elastic ice plate floats on incompressible inviscid fluid of infinite depth. The solution is obtained by multipole expansion method.

2. Mathematical formulation

Consider the problem of a submerged sphere of radius a advancing at constant forward speed U. We define a Cartesian coordinate system $O-xyz$ so that the upper undisturbed water surface is $z=0$ and z points upwards. The system is moving with the sphere at the same speed. The centre of the sphere is located at $x=y=0,\ z=-h\ (h>0)$. We also define a spherical coordinate system (r,θ,β) with the origin fixed at the position of the centre of the sphere. These two systems are related by the following equations:

$$x = r\sin\theta\cos\beta, \quad y = r\sin\theta\sin\beta, \quad z = r\cos\theta - h.$$

The ice sheet is treated as a thin elastic plate with the lateral stress. We assume that the fluid motion beneath the ice is irrotational and can be described by a velocity potential $\Phi(x,y,z) = -U[x-\phi(x,y,z)]$, where ϕ is the steady potential due to unit forward speed. In a frame of reference moving with the sphere speed in the positive x-direction, the equation for the small vertical deflection $\zeta(x,y)$ of a thin plate floating on water is

$$D\Delta_2^2\zeta + Q\Delta_2\zeta + MU^2\frac{\partial^2\zeta}{\partial x^2} - \rho U\frac{\partial\phi}{\partial x} + \rho g\zeta = 0 \quad (z=0), \tag{1}$$

where

$$D = Eh_1^3/[12(1-\nu^2)], \quad M = \rho_1 h_1, \quad \Delta_2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

E is the Young's modulus for the ice, ν is its Poisson's ratio, Q is the lateral stress (compression for $Q>0$ and stretch for $Q<0$), ρ_1 is the density of the ice, h_1 is the thickness of the ice-cover, ρ is the density of water, and g is the acceleration due to gravity. When the flexural rigidity D and the compressive force Q are taken to be zero, the ice sheet behaves as a floating set of disconnected mass points (the broken ice). When in addition also surface density of ice-cover M is taken to be zero, then upper boundary of fluid becomes a free-surface.

The velocity potential $\phi(x,y,z)$ should satisfy the Laplace equation in the fluid domain

$$\Delta\phi = 0 \quad (-\infty < x,y < \infty,\ z<0).$$

The kinematic condition at the ice-water interface is

$$\partial\zeta/\partial x = -\partial\phi/\partial z \quad (z=0). \tag{2}$$

The boundary condition on the body surface S is

$$\partial\phi/\partial n = n_x \quad (r=a), \tag{3}$$

where $\mathbf{n}$ is the inward normal of the body surface and n_x its component in x direction. The radiation condition assumes that only those outgoing wave with group velocity larger than forward speed can be found in far front of the body.

We write the steady potential in terms of the following multipole expansion based on the Legendre functions P_n^m (Wu, 1995)

$$\phi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_n^m \left[\frac{a^{n+1}}{r^{n+1}} P_n^m(\cos\theta) \cos m\beta + F_n^m(r,\theta,\beta) \right], \tag{4}$$

where the first term in the square brackets in (4) is for the sphere in an unbounded fluid domain and the second term is introduced to satisfy the conditions (1), (2) on the upper water surface

$$F_n^m = \frac{a^{n+1} i^m}{2\pi(n-m)!} \int_L \int_{-\pi}^{\pi} A(k,\gamma) k^n \cos m\gamma e^{kz} e^{ik(x\cos\gamma + y\sin\gamma)} d\gamma dk, \tag{5}$$

where

$$A(k,\gamma) = e^{-kh} T(k,\gamma)/Z(k,\gamma), \quad T(k,\gamma) = Z(k,\gamma) + 2\rho U^2 k \cos^2\gamma,$$
$$Z(k,\gamma) = Dk^4 - Qk^2 - U^2 k \cos^2\gamma(\rho + Mk) + \rho g. \tag{6}$$

The integration route L in (5) is from zero to infinity. Under certain constrains on the compressive force, the equation $Z(k,\gamma) = 0$ has two real positive roots k_1 and k_2 ($k_1 < k_2$) only at $U|\cos\gamma| > U_m$, where the speed U_m is the minimum phase velocity of the flexural-gravity waves. It is well known (see, for example, Squire *et al.*, 1996), that the group velocity of these waves exceeds the phase velocity at shorter wavelengths (large wave numbers), but is less than the phase speed at longer wavelengths (smaller wave numbers). The phase velocity and the group velocity coincide at the minimum phase velocity U_m - often called the *critical velocity* - which is equal to

$$U_m = \sqrt{\frac{Dk_0^4 - Qk_0^2 + \rho g}{k_0(\rho + Mk_0)}}, \tag{7}$$

where the *critical wave number* k_0 is the real positive root of the equation

$$Dk_0^4(2Mk_0/\rho + 3) - Qk_0^2 - 2Mgk_0 - \rho g = 0. \tag{8}$$

We can neglect the inertial force due to mass of the ice sheet since it is much less than the inertial force due to fluid motion. Under this assumption $M = 0$ and k_0 can be obtained from (8) as

$$k_0^2 = \frac{Q + \sqrt{Q^2 + 12\rho g D}}{6D}.$$

At $Q = 0$, we have from this solution (Yeung & Kim, 1998)

$$k_0 = \left(\frac{\rho g}{3D}\right)^{1/4}. \tag{9}$$

For small compressive force there is a nearly linear decrease with respect to Q (Shulkes *et al.*, 1987)

$$U_m \approx U_0\left(1 - \frac{3}{4}\varepsilon\right), \quad U_0 = 2\left(\frac{Dg^3}{27\rho}\right)^{1/8}, \quad \varepsilon = \sinh^{-1}\frac{Q}{\sqrt{12\rho g D}}, \tag{10}$$

where U_0 is the minimum phase velocity at $M = Q = 0$.

The path at the singularities in the integrand of (5) is determined by the radiation condition. Short-wavelength elastic waves have a group velocity greater than their phase velocity and propagate ahead of the sphere, whereas the longer gravity waves propagate behind. As a consequence when

$|\gamma| < \pi/2$, L passes over the singularity k_1 and under the singularity k_2. And vice versa, when $|\gamma| > \pi/2$, L passes under the singularity k_1 and over the singularity k_2. This leads to

$$\phi = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m \Bigg[\frac{a^{n+1}}{r^{n+1}} P_n^m(\cos\theta)\cos m\beta -$$

$$\frac{a^{n+1}}{2\pi(n-m)!}\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'} \varepsilon_{m'} \frac{r^{n'}}{(n'+m')!} P_{n'}^{m'}(\cos\theta)\cos m'\beta\; I(m,n,m',n')\Bigg],$$

where $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ if $m > 0$,

$$I(m,n,m',n') = -(-1)^{\frac{m-m'}{2}} 4 \int_0^{\pi/2} pv \int_0^{\infty} k^{n+n'} \cos m\gamma \cos m'\gamma e^{-2kh} \frac{T(k,\gamma)}{Z(k,\gamma)} dk d\gamma$$

if $m - m'$ is even and

$$I(m,n,m',n') = -(-1)^{\frac{m-m'-1}{2}} 4\pi \int_0^{\gamma_0} \cos m\gamma \cos m'\gamma \Bigg[k_1^{n+n'} e^{-2k_1 h} \frac{T(k_1,\gamma)}{Z'(k_1,\gamma)} -$$

$$k_2^{n+n'} e^{-2k_2 h} \frac{T(k_2,\gamma)}{Z'(k_2,\gamma)} \Bigg] d\gamma \qquad (11)$$

if $m - m'$ is odd. Here pv indicates the principal-value integration, $Z'(k_j,\gamma) = \partial Z/\partial k|_{k=k_j}$ $(j = 1,2)$, γ_0 is defined as

$$\gamma_0 = \begin{cases} 0 & U < U_m, \\ \arccos(U_m/U) & U > U_m. \end{cases} \qquad (12)$$

By applying the condition on the body surface (3) and using orthogonality relations for associated Legendre functions, we obtain the infinite system of linear equations

$$\frac{n+1}{a} A_n^m + \frac{n\varepsilon_m}{2\pi} \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'} \frac{a^{n+n'}}{(n+m)!(n'-m')!} I(m',n',m,n) A_{n'}^{m'} = \delta_{n1}\delta_{m1}$$

for the unknown coefficients A_n^m, where δ_{ij} is the Kronecker delta function. This system can be solved numerically by truncating it to $N \times N$ system and increasing N until the solution converges to the required degree of accuracy.

Once the solution is found, the hydrodynamic load may be determined by integrating the pressure obtained from the Bernoulli equation over the body surface

$$F_j = -\frac{1}{2}\rho U^2 \int_S \nabla(\phi - x)\nabla(\phi - x) n_j dS, \quad (j = 1,2,3), \quad (n_1, n_2, n_3) = (n_x, n_y, n_z),$$

while the moment about the centre of the sphere is apparently zero. Following the derivation by Wu (1995), we have

$$F_1 = -2\rho\pi U^2 \Bigg[\sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{1}{\varepsilon_m} \frac{n+2}{n+1} \frac{(n+m+2)!}{(n-m)!} A_n^m A_{n+1}^{m+1} - \sum_{n=2}^{\infty}\sum_{m=0}^{n-2} \frac{1}{\varepsilon_m} \frac{n+1}{n} \frac{(n+m)!}{(n-m-2)!} A_n^m A_{n-1}^{m+1} \Bigg],$$

$$F_3 = 4\rho\pi U^2 \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{1}{\varepsilon_m} \frac{n+2}{n+1} \frac{(n+m+1)!}{(n-m)!} A_n^m A_{n+1}^m.$$

As consequence of the symmetry, we have $F_2 = 0$.

For broken ice ($D = Q = 0$), the equation $Z(k,\gamma) = 0$ in (6) has only one real positive root k_1 at all values of the speed U

$$k_1 = \frac{\sqrt{\rho(\rho + 4gM\sec^2\gamma/U^2)} - \rho}{2M}.$$

For free surface ($M = 0$), this root is equal to $k_1 = g \sec^2 \gamma / U^2$. In these two cases, the phase velocity of the wave exceeds the group velocity and the gravity waves propagate behind the sphere. When $|\gamma| < \pi/2$, L passes over the singularity in (5) and when $|\gamma| > \pi/2$, L passes under the singularity. As this takes place, the second term in the square brackets in (11) should be omitted and the value γ_0 in (12) is equal to $\pi/2$.

3. Numerical results

Numerical calculations are performed for the following input data:

$$E = 5 \times 10^9 Pa,\ \rho = 1025\ kg\ m^{-3},\ \rho_1 = 922.5\ kg\ m^{-3},\ \nu = 0.3,\ a = 10m.$$

The ice thickness h_1 and magnitude Q characterizing the ice compression (stretch) changed between limits $0.5m \le h_1 \le 2m$ and $-1.5\sqrt{D} \le Q/\sqrt{\rho g} \le 1.5\sqrt{D}$.

Fig. 1(a) shows the values of critical wave number $k_0 a$ calculated from the equation (8) as a function of the lateral stress. The symbols $1, 2, 3$ denote the value $k_0 a$ in (9) for $h_1 = 0.5; 1; 2m$, respectively. In Fig. 1(b) are presented the critical velocities $U_m/\sqrt{ga}$ given by (7) (solid lines) and by (10) (dashed lines).

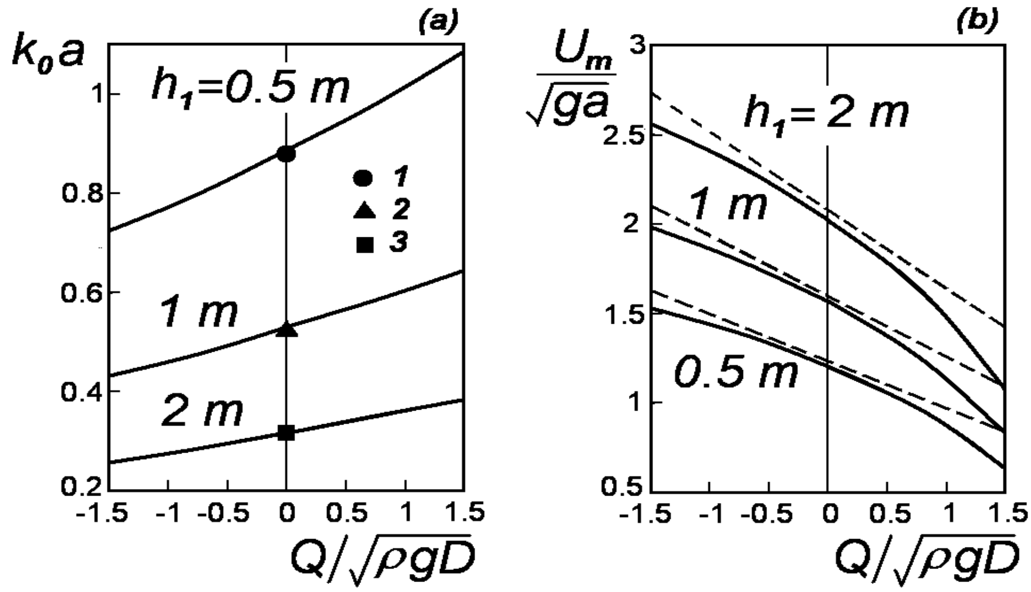

Figure 1.

More detailed results for the hydrodynamic load (wave resistance and lift) and the ice deflection over the sphere will be presented at the Workshop.

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