Cloaking of a cylinder in waves

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1. Introduction

This paper examines the process of making a vertical cylinder of uniform circular cross section appear invisible to incident waves in the sense that the observer in the far field sees only the incident wave field and no signature of the vertical cylinder in the form of circular diffracted waves.

Such a process is referred to in the literature as 'cloaking' and the concept was first demonstrated by Pendry *et al.* (2006) in the context of electromagnetic wave scattering, a paper which has since been cited over 1200 times. The idea proposed in Pendry *et al.* (2006) is to surround a cylindrical cavity by an annular 'cloaking' region of material which has continuous varying materials properties (permeability and permittivity) thereby effecting a continuously varying refractive index in the cloaking region. The connection of the outer boundary of the cloaking region with free space is also shown to form a so-called PML (perfectly matched layer) implying that the cloak itself is reflectionless, whilst no fields generated inside the cylindrical void can escape through the cloak. These theoretical results are illustrated by taking a ray-theory limit and plotting rays which are shown to bend around the cylinder within the annular cloaking region. Thus, in that illustration, rays in front of the cylinder appear to have come from behind the cylinder. Some simple arguments show that the material parameters must be dispersive with frequency and so a perfect cloak can only be designed to work at a single frequency. It is natural to ask how effective such a cloak is over a range of frequencies.

The key element of the approach of Pendry *et al.* (2006) is the mapping of the domain with the cylindrical inclusion into the whole space. Crucially, the transformation preserves Maxwell's equations (Ward & Pendry (1996)) provided the material parameters, permeability and permittivity, are rescaled in the appropriate manner. This rescaling is particularly simple if the transformation is angle-preserving. Thus the cylinder is formed by the inverse mapping of a cut in the transformed plane in which the solution of waves propagating uniformly with constant material properties is imposed (i.e. a solution in which there is no scattering). The varying material properties in the annular cloaking region are then determined simply by the inverse mapping of constant material parameters.

In the context of electromagetics, Maxwell's equations are vector equations and this, it seems, is crucial in allowing the transformation of coordinates to preserve them.

In contrast, in linear water wave theory, the governing equations are scalar, and the only sensible 'material parameter' we have the flexibility to alter in an annular region surrounding a cylinder is the depth of the bed. However, we recognise that changes in depth allow us to design a spatially-varying 'refractive index'. Thus the idea being proposed in this paper is that we use changes in depth in an annular region surrounding the vertical cylinder to render the cylinder invisible to incident waves in the far field.

We must briefly mention a different approach to cloaking in linear water wave theory that has recently been presented by Farhat *et al.* (2008), in which an annular region populated by a large number of vertical posts of small cross-section has been used to bend waves around a cylinder. In that work, homogenisation methods are used to argue that the large cylinder array in the cloaking region forms an effective medium with an anisotropic refractive index. Direct numerical simulations, as well as experiments, are used to confirm that cloaking can occur. However, this cloaking mechanism works well because of a separation of scales; waves are very short compared to the cylinder being cloaked, but long compared to each cylindrical element of the cloak which allows homogenisation techniques to succeed. Indeed, the short wavelengths involved necessitates the inclusion of capilliarity in their work.

2. Governing equations

We assume the use of cylindrical polar coordinates (r, θ, z). An impermeable vertical cylinder of constant circular cross section centred along the z-axis and radius a extends through the depth. The sea-bed is given by $z = -h(r, \theta)$ for $r > a$ where $h(r, \theta)$ is a continuous function with continuous derivatives and is such that $h(r, \theta) = h_0$, a constant, for $r > b$.

Using linearised water wave theory a velocity potential is given by $\Re\{\Phi(r, \theta, z)\mathrm{e}^{-\mathrm{i}\omega t}\}$ where ω is the assumed angular frequency of motion. Then Φ satisfies

$$(\nabla^2 + \partial_{zz})\Phi = 0, \quad -h(r, \theta) < z < 0, \; r > a, \quad (1)$$

where $\nabla = (\partial_r, r^{-1}\partial_\theta)$,

$$\Phi_z - \nu\Phi = 0, \quad \text{on } z = 0, \; r > a, \quad (2)$$

where $\nu = \omega^2/g$, g is gravitational acceleration and

$$\Phi_z + \nabla h.\nabla\Phi = 0, \quad \text{on } z = -h(x, y), \; r > a, \quad (3)$$

which reduces to $\Phi_z = 0$, on $z = -h_0$ for $r > b$. On the cylinder we have

$$\Phi_r(a, \theta) = 0, \quad -h(a, \theta) < z < 0, \; -\pi < \theta < \pi.$$

Since the cylinder geometry is symmetric we only need consider an incident wave propagating in the direction $\theta = 0$, given by the potential

$$\Phi_{inc}(r, \theta, z) = \mathrm{e}^{\mathrm{i}k_0 r\cos\theta} f(k_0 h_0, k_0 z),$$

where

$$f(kh, kz) = \frac{\cosh(kh + kz)}{\cosh kh}, \quad (4)$$

and $k = k_0$ is the real positive root corresponding to $h = h_0$ of

$$k \tanh kh = \nu. \quad (5)$$

Consequently, we let the bed symmetric about the x-axis so that $h(r, \theta) = h(r, -\theta)$. The total potential is $\Phi = \Phi_{inc} + \Phi_{sc}$ where Φ_{sc} is a symmetric scattered potential which, on account of the radiation condition, satisfies

$$\Phi_{sc} \sim \mathcal{A}(\theta)\sqrt{\frac{2}{\pi k_0 r}}\,\mathrm{e}^{\mathrm{i}(k_0 r - \pi/4)} f(k_0 h_0, k_0 z),$$

as $k_0 r \to \infty$ where $\mathcal{A}(\theta) = \mathcal{A}(-\theta)$ is the diffraction coefficient.

The requirement for a cylinder to be cloaked is that $A(\theta) = 0$ for $0 < \theta < \pi$. Alternatively, the total energy scattered to infinity, measured by

$$\mathcal{E} = \frac{1}{\pi}\int_0^\pi |\mathcal{A}(\theta)|^2 \mathrm{d}\theta, \quad (6)$$

must be zero. It is well known that $\mathcal{E} \equiv -\Re\{\mathcal{A}(0)\}$.

If $h(r, \theta) = h_0$ for all $r > a$ so the bed is flat everywhere, then the exact solution is well-known (McCamy & Fuchs) and given by

$$\Phi_{cyl} = \psi_{cyl}(r, \theta) f(k_0 h_0, k_0 z), \quad (7)$$

where

$$\psi_{cyl} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n \left(J_n(k_0 r) - Z_n H_n(k_0 r)\right) \mathrm{e}^{\mathrm{i}n\theta}, \quad (8)$$

with $Z_n = J_n'(k_0 a)/H_n'(k_0 a)$ and then

$$\mathcal{A}_{cyl}(\theta) = -\sum_{n=-\infty}^{\infty} Z_n \mathrm{e}^{\mathrm{i}n\theta}, \quad (9)$$

and the total scattered wave energy equates to

$$\mathcal{E}_{cyl} = \sum_{n=-\infty}^{\infty} |Z_n|^2 \equiv \Re\left\{\sum_{n=-\infty}^{\infty} Z_n\right\}, \quad (10)$$

which is never zero, as expected.

3. Transformation

It is worth sketching out how the transformation method of Pendry *et al.* (2006) might apply to the theory of linearised water waves. Thus, we assume a conformal transformation of the horizontal (x, y) coordinates into a new coordinate system (u, v) via a mapping $\xi = f(\zeta)$ where $\xi = u + \mathrm{i}v$ and $\zeta = x + \mathrm{i}y$. In order to preserve the three-dimensional Laplace's equation, we are required to scale the vertical coordinate to $w = |f'|z$ and the bottom boundary $z = h(x, y)$ is mapped to $w = H(u, v) = |f'|h(x, y)$. The transformation f is designed to map the domain including the cylinder into the whole (u, v) plane. A simple example of such a mapping is $f(\zeta) = \zeta + a^2/\zeta$; but this is not the only one and the variation of this used by Pendry *et al.* (2006) could equally well be employed here. Then (1) is preserved under this transformation,

$$(\tilde{\nabla}^2 + \partial_{ww})\tilde{\Phi} = 0, \quad (11)$$

with $\tilde{\nabla} = (\partial_u, \partial_v)$ and $\tilde{\Phi}(u, v, w) \equiv \Phi(x, y, z)$ whilst the free surface condition (2) becomes

$$|f'|\tilde{\Phi}_w - \nu\tilde{\Phi} = 0, \quad w = 0, \quad (12)$$

and the bed condition (3) maps to

$$\tilde{\Phi}_w + \tilde{\nabla}H.\tilde{\nabla}\tilde{\Phi} + |f'|H\tilde{\nabla}(|f'|^{-1}).\tilde{\nabla}\tilde{\Phi} = 0, \quad (13)$$

on $w = H$. In this transformed problem in which the cylinder has been removed, cloaking now requires that, for a given mapping f, a function H exists such that the solution on (11)–(13) subject to an incident wave from infinity scatters no waves to infinity. If this can be achieved, then a cloaking topography h surrounding the cylinder in the physical plane can be recovered by the inverse mapping.

Notice the presence of spatially-varying coefficient $|f'|$ in the transformed free-surface and bed conditions (12) and (13). It seems that unlike the approach of Pendry *et al.* (2006), the equations cannot be made invariant to the transformation.

Thus, we return to the original problem specification and attack it directly.

4. The mild-slope approximation

The three-dimensional linearised water wave problem is approximated by employing the mild-slope method. That is, assuming the depth-dependence assigned to propagating modes over a locally-flat bed

$$\Phi(r, \theta, z) \approx \phi(r, \theta) f(kh, kz), \quad (14)$$

in which $k(h(r,\theta))$ denotes the positive, real root of (5) where the depth is $h(r,\theta)$. We follow Chamberlain & Porter's (1995) implementation of the approximation (14) which uses a variational principle to replace (1), (2) and (3) by the single modified mild-slope equation.

After a transformation into its canonical form, achieved by writing

$$\phi(r,\theta) = \{u_0(h_0)/u_0(h(r,\theta))\}^{1/2}\psi(r,\theta), \quad (15)$$

where $u_0 = \mathrm{sech}^2 kh(2kh + \sinh 2kh)/(4k)$ then ψ can be shown to satisfy

$$\nabla^2\psi + \kappa(r,\theta)\psi = 0, \quad r > a, \quad (16)$$

where

$$\kappa(r,\theta) = k^2 + A\nabla^2 h + B(\nabla h)^2, \quad (17)$$

and, with the abbreviation $K = 2kh$,

$$\begin{aligned} A &= -2k/(K + \sinh K), \\ B &= k^2\{K^4 + 4K^3 \sinh K + 3K^2(2\cosh^2 K + 1) \\ &\quad + 18K \sinh K + 3\sinh^2 K(2\cosh K + 5)\}/ \\ &\quad \{3(K + \sinh K)^4\}. \end{aligned}$$

Finally we mimic the decomposition of Φ writing $\psi(r,\theta) = \psi_{inc}(r,\theta) + \psi_{sc}(r,\theta)$ where

$$\psi_{inc} = \mathrm{e}^{\mathrm{i}k_0 r\cos\theta}, \quad (18)$$

and

$$\psi_{sc} \sim \mathcal{A}(\theta)\sqrt{\frac{2}{\pi k_0 r}}\,\mathrm{e}^{\mathrm{i}(k_0 r - \pi/4)}, \quad (19)$$

whilst $\psi_r(a,\theta) = 0$.

We now follow closely the method described in Griffiths & Porter (2011), reformulating the wave equation (16) into an integral equation with the use of Green's Identity applied to $\psi - \psi_{cyl}$ and a Green's function $G(r, r'; \theta, \theta')$ satisfying

$$(\nabla^2 + k_0^2)G = r\delta(r - r')\delta(\theta - \theta'), \quad (20)$$

and

$$G_r(a, r'; \theta, \theta') = 0, \quad (21)$$

and it can readily be shown that

$$\begin{aligned} G &= -\frac{\mathrm{i}}{4}H_0(k_0\rho) \\ &\quad + \frac{\mathrm{i}}{4}\sum_{n=-\infty}^{\infty} Z_n H_n(k_0 r) H_n(k_0 r')\mathrm{e}^{\mathrm{i}n(\theta-\theta')}, \end{aligned}$$

where $\rho^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta')$. The result of this procedure is

$$\begin{aligned} \iint_D [\kappa(r,\theta) - k_0^2] G(r,\theta; r',\theta')\psi(r,\theta)\, r\mathrm{d}r\mathrm{d}\theta \\ + \psi(r',\theta') = \psi_{cyl}(r',\theta'), \quad (22) \end{aligned}$$

where ψ_{cyl} is defined by (8). Thus (22) serves as an integral equation for the unknown ψ when restricted to $D := \{a < r < b, \ -\pi < \theta < \pi\}$, the region of varying topography and defines ψ beyond D once ψ is known in D.

Taking $kr' \to \infty$ allows us to access the far-field behaviour of ψ which, in comparison with (19), gives

$$\begin{aligned} \mathcal{A}(\theta') &= \mathcal{A}_{cyl}(\theta') \\ &\quad + \frac{\mathrm{i}}{4}\iint_D [\kappa(r,\theta) - k_0^2]\psi_{cyl}(r,\theta')\psi(r,\theta)\, r\mathrm{d}r\mathrm{d}\theta, \quad (23) \end{aligned}$$

where $\mathcal{A}_{cyl}$ is defined by (9).

The free surface elevation due to an incident wave of unit amplitude is given by $\eta(r,\theta) = \Phi(r,\theta,0)$ which can be accessed from (14), (15).

5. Results

We are interested in the 'cloaking factor'

$$C = \frac{\mathcal{E}}{\mathcal{E}_{cyl}}, \quad (24)$$

where $\mathcal{E}_{cyl}$ defined by (10) is the total scattered energy in the absence of a cloaking region, D, and $\mathcal{E}$ is defined by (6) with (23) for h varying in D. When $C < 1$, the cylinder with the cloaking region containing the variable bed scatters less energy in circular waves than with a flat bed. Perfect cloaking requires $C = 0$.

The solution to the integral equation (22) is approximated numerically using the method described in Griffiths & Porter (2011), and this provides a numerical approximation to $\mathcal{A}$ in (23).

The varying bed $h(r, \theta)$ is defined in a Fourier basis with

$$h(r, \theta) = h_0 + \sum_{p=1}^{P} \sum_{q=0}^{Q-1} \alpha_{pq} f_p(r) \cos(2q\theta), \quad (25)$$

where

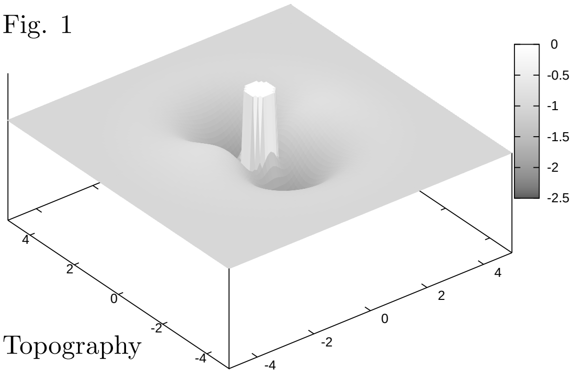
$$f_p(r) = T_{2p}\left(\frac{b-r}{b-a}\right) - (-1)^p, \qquad (26)$$

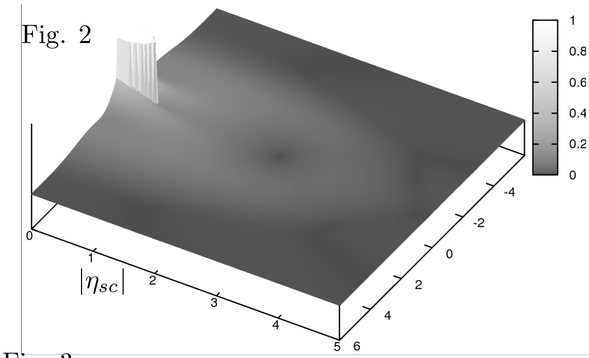
and $T_n(x)$ are Chebychev polynomials such that $h(b, \theta) = h_0$ and $h_r(b, \theta) = 0$.

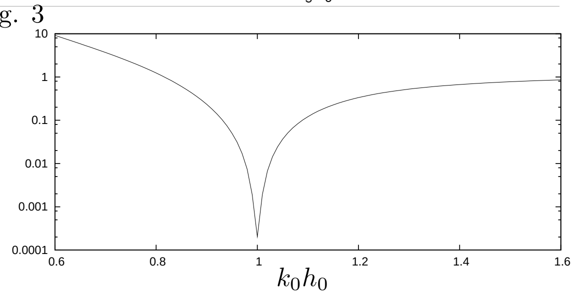
In (25) we have PQ unknown weighting coefficients α_{pq} and these are used as free variables in an optimisation procedure to minimise C, initialised with $\alpha_{pq} = 0$ for all p, q (i.e. a flat-bed), where $C = 1$.

Numerically it has been shown that, for certain ranges of wavenumbers and cylinder ratios a/h_0, C tends to zero for modest values of P and Q. One such example is given in the figures opposite defined by the parameters $k_0h_0 = 1$, $a/h_0 = \frac{1}{2}$, $b/a = 10$ and $P = 3$, $Q = 2$ (i.e. just 6 degrees of freedom in the definition of the bed) which results numerically in a minimisation to $C = 10^{-4}$ (figure 3). In figure 1, the topography defined for cloaking is illustrated and figure 2 shows the resulting amplitude of the scattered free surface when a wave is incident on the topography. It can be seen that the scattered wave does indeed decay rapidly away from the cylinder with no waves radiated to infinity. Figure 3, shows C as a function of wavenumber k_0h_0 for the topography defined for cloaking at $k_0h_0 = 1$. Further results will be presented at the workshop.

The minimisation of C to zero is robust to changes in the accuracy of the numerical scheme. Of course, these results have been obtained under the mild-slope approximation and further work would be needed to compute accurate solutions to the full three-dimensional linear wave equations to confirm the cloaking phenomenon.







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7. References

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