Band structures and band gaps in water-wave scattering by periodic lattices of arbitrary bodies

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1 Introduction

In standard wave refraction, the directions of the incident and refracted waves lie on different sides of the normal direction of the interface between two media. Snell's law states that the ratio of the sines of the angles of incidence and refraction is equal to the opposite ratio of the indices of refraction. If the refraction index is negative, then the refracted wave lies on the same side of the normal as the incident wave. Materials with negative refractive index have very interesting properties although they do not occur naturally. For example, they can be used as a so-called superlens: a point source placed in front of a flat rectangular slab that exhibits negative refraction is refocused to form a real image on the opposite side of the slab.

It is well known that negative refraction of electromagnetic waves can occur in metamaterials or photonic crystals and this has been quite well studied (for example, see Botten *et al.*, 2001; Joannopoulos *et al.*, 1995, and references therein). It was shown only recently, both theoretically and experimentally, that negative refraction can also occur in water-wave diffraction (Hu *et al.*, 2004). The "material" having negative refractive index was an arrangement of 202 periodically distributed bottom-mounted circular cylinders. Further investigations were made by Li & Mei (2007) and Farhat *et al.* (2010), all of which make use of periodic distributions of bottom-mounted circular cylinders to make up the effective scatterer. A natural question to ask is whether periodic arrangements of other types of scatterers also exhibit properties which lead to negative refraction.

The main tool for analysing whether a material allows a negative refractive index is its band structure. Besides this, the band diagram can be considered a very useful way to represent the propagation properties of a medium in general. In order for a material to allow negative refraction (for some frequencies) it has to exhibit a band gap, i.e. there exists a range of frequencies, for which no wave can pass through the medium in any direction without loss of amplitude (note that this alone does not guarantee negative refractive index, however; see Hu *et al.*, 2004, for further information). Thus, it is necessary to be able to compute the band structure of periodic arrangements of bodies. For bottom-mounted circular cylinders, a method was described by McIver (2000) although he did not use it to compute the type of band diagrams required for the investigation of negative refraction. On the other hand, a method for computing the passing and stopping bands of lattices of arbitrary bodies was derived in Peter & Meylan (2009*a*), which is based on work presented at a recent workshop (Peter & Meylan, 2009*b*), as a side result. It turns out that this method can be modified to compute the band diagrams necessary for the investigation of negative refraction of arrangements of arbitrary bodies.

After stating the problem, we briefly review McIver's method before discussing how the method of Peter & Meylan (2009*a*) can be modified to compute the required band structures. Some preliminary simulation results comparing the methods for the case of bottom-mounted cylinders are also shown and they give convincing agreement.

2 Statement of the problem

We consider water-wave scattering by a periodic lattice of vertically non-overlapping bodies. The equations of motion for the water are derived from the linearised inviscid theory assuming irrotational motion. Restricting to time-harmonic motion with radian frequency ω (which is the spectral parameter), the velocity potential Φ can be expressed as the real part of a complex quantity, $\Phi(\mathbf{y},t) = \mathbb{R}\mathrm{e}\,\{\phi(\mathbf{y})\mathrm{e}^{-\mathrm{i}\omega t}\}$. To simplify notation, $\mathbf{y} = (x, y, z)$ always denotes a point in the water, which is assumed to be of constant finite depth d, while $\mathbf{x} = (x, y)$ always denotes a point of the undisturbed water surface assumed at $z = 0$, i.e. $\mathbf{x} = (x, y, 0)$ is meant when appropriate.

Writing $\alpha = \omega^2/g$, where g is the acceleration due to gravity, the potential ϕ has to satisfy the standard boundary-value problem

$$-\Delta\phi = 0, \qquad \mathbf{y} \in D, \tag{1}$$

$$\frac{\partial\phi}{\partial z} = \alpha\phi, \qquad \mathbf{x} \in \Gamma^{\mathrm{f}}, \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 0, \qquad \mathbf{y} \in D,\ z = -d, \tag{3}$$

where D is the domain occupied by the water and Γ^{f} is the free water surface. At the immersed body surface, the water velocity potential has to equal the normal velocity of the body. A further relationship between the potential and its normal derivative on the body surface is required if the velocity depends on the potential, and this comes from the equation of motion for the body. For future reference, we note that the positive wavenumber k is related to α by the dispersion relation

$$\alpha = k \tanh kd. \tag{4}$$

2.1 Restriction to a periodicity cell

We assume that the geometry of the lattice is such that it consists of repeated copies of a cuboid unit cell and we adopt the notation of McIver (2000) for the description of the lattice. We begin with the general definition of the lattice in order to point out the relation to photonics (see Joannopoulos *et al.*, 1995, for details).

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be two (two-dimensional) vectors that span the lattice: that is every translation between the mean-centre position of bodies in the horizontal plane has the form of a *lattice vector*

$$\mathbf{R} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2, \tag{5}$$

where $m_1, m_2 \in \mathbb{Z}$. The corresponding *reciprocal lattice vectors* $\mathbf{K}$ satisfy

$$\mathbf{K} \cdot \mathbf{R} = 2\pi p, \tag{6}$$

where $p \in \mathbb{Z}$. If the reciprocal lattice vectors are written as

$$\mathbf{K} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2 \tag{7}$$

for $n_1, n_2 \in \mathbb{Z}$, (6) is satisfied provided that

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}, \tag{8}$$

where δ_{ij} is the Kronecker delta.

Bloch's theorem justifies looking for solutions of the form

$$\phi(\mathbf{y} + (\mathbf{R}, 0)) = \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{R}}\phi(\mathbf{y}), \tag{9}$$

for all lattice vectors $\mathbf{R}$. The real part of $\mathbf{q}$ measures the change in the phase from one point in the lattice to its counterparts in the neighbouring periodicity cells while the imaginary part encodes the change in amplitude as a wave propagates through the array.

In our case, where the unit cell is a cuboid of length L, width W (and height d), we have $\mathbf{a}_1 = L\mathbf{i}$ and $\mathbf{a}_2 = W\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in two-dimensional space. The corresponding primitive reciprocal lattice vectors are $\mathbf{b}_1 = \frac{2\pi}{L}\mathbf{i}$ and $\mathbf{b}_2 = \frac{2\pi}{W}\mathbf{j}$. For this geometry, (9) is equivalent to the four independent (Bloch) conditions

$$\phi(L/2, y, z) = \mathrm{e}^{\mathrm{i}q_1 L}\phi(-L/2, y, z), \quad y \in (-W/2, W/2),\ z \in (-d, 0), \tag{10}$$

$$\partial_x\phi(L/2, y, z) = \mathrm{e}^{\mathrm{i}q_1 L}\partial_x\phi(-L/2, y, z), \quad y \in (-W/2, W/2),\ z \in (-d, 0), \tag{11}$$

$$\phi(x, W/2, z) = \mathrm{e}^{\mathrm{i}q_2 W}\phi(x, -W/2, z), \quad x \in (-L/2, L/2),\ z \in (-d, 0), \tag{12}$$

$$\partial_y\phi(x, W/2, z) = \mathrm{e}^{\mathrm{i}q_2 W}\partial_y\phi(y, -W/2, z), \quad x \in (-L/2, L/2),\ z \in (-d, 0). \tag{13}$$

Equation (9) is unchanged if a reciprocal lattice vector $\mathbf{K}$ is added to $\mathbf{q}$. Thus, given a solution $\phi(\mathbf{y}; \mathbf{q})$ then $\phi(\mathbf{y}; \mathbf{q} + \mathbf{K})$ is also a solution. Consequently, it is sufficient to restrict attention to the *first Brillouin zone* $\{\mathbf{q} \,|\, \mathrm{Re}\, q_1 \in (-\pi/L, \pi/L],\ \mathrm{Re}\, q_2 \in (-\pi/W, \pi/W)\}$.

3 Variational formulation for bottom-mounted cylinders

In case of the scatterers being rigid bottom-mounted circular cylinders of cross-sectional area A, a very simple solution method was suggested by McIver (2000). In this case, the depth dependence can be factored out of the problem, i.e. we look for a potential $u(\mathbf{x})$ only dependent on the horizontal coordinates satisfying the Helmholtz equation

$$-\Delta u(\mathbf{x}) = k^2 u(\mathbf{x}) \quad \text{in } \tilde{D}, \tag{14}$$

where $\tilde{D} = (-L/2, L/2) \times (-W/2, W/2) \backslash \overline{A}$ subject to a homogeneous Neumann condition on the boundary of the cylinder cross-section and (9). The potential ϕ is $\phi = u(\mathbf{x}) \cosh k(z + d)$.

For a specified real Bloch wave vector $\mathbf{q} = (q_1, q_2)$ this problem is self-adjoint and the corresponding infinite sequence of eigenvalues $\lambda = k^2$ (note that the problem is stated on a finite domain) of the negative Laplacian may be determined by the Rayleigh–Ritz method. The Rayleigh quotient for a given trial function u is

$$\mathcal{R}(u) = \frac{\int_{\tilde{D}} |\nabla u|^2 \, \mathrm{d}\mathbf{x}}{\int_{\tilde{D}} |u|^2 \, \mathrm{d}\mathbf{x}}. \qquad (15)$$

In order for u to satisfy the Bloch condition (9), we expand u as

$$u = \sum_{m,n\in\mathbb{Z}} C_{mn} \mathrm{e}^{\mathrm{i}(\mathbf{q}+\mathbf{K}_{mn})\cdot\mathbf{x}}, \qquad (16)$$

where $\mathbf{K}_{mn} = 2\pi\left(\frac{m}{L}\mathbf{i} + \frac{n}{W}\mathbf{j}\right)$ is a reciprocal lattice vector, as defined in (6). Approximations to the eigenvalues correspond to the local minima of $\mathcal{R}(u)$ with respect to variations in the coefficients C_{mn}. This leads to the generalised eigenvalue problem

$$\sum_{m,n\in\mathbb{Z}} (\mathcal{E}_{klmn} - \lambda\mathcal{H}_{klmn})C_{mn} = 0, \quad k,l \in \mathbb{Z}, \quad (17)$$

where the elements of the second-order tensors $\mathcal{E}_{klmn}$ and $\mathcal{H}_{klmn}$ can be calculated explicitly thanks to the simple cylindrical geometry (see McIver, 2000, for details).

4 Lattices of arbitrary bodies

The problem of water-wave scattering by periodic line arrays of arbitrary bodies was solved by Peter *et al.* (2006) and it was briefly pointed out in Peter & Meylan (2009*a*) that this method can be used to calculate passing and stopping bands for doubly periodic arrays (lattices) of bodies, which is considered here. We briefly recall these ideas and show how the method can be modified for computing the band structures. Note that, in particular, the method is not restricted to periodic lattices of single structures. (It was shown in Peter & Meylan (2009*a*) how multi-body structures can be handled.)

4.1 Scattering by a periodic line array

First, we summarise how the far field can be calculated, which describes the scattering of a plane incident wave of potential ϕ^{In} of wavenumber k and incident angle χ far away from the line array. We define the scattering angles, which give the directions of propagation of plane scattered waves far away from the array. Letting $p = 2\pi/L$, the scattering angles χ_m are

$$\chi_m = \arccos(\psi_m/k), \text{ where } \psi_m = k\cos\chi + mp,$$

and we write ψ for ψ_0. Also note that $\chi_0 = \chi$ by definition. If $|\psi_m| < k$, we say that $m \in \mathcal{M}$ and then $0 < \chi_m < \pi$.

It turns out that, as $y \to \pm\infty$, the far field consists of a set of plane waves propagating in the directions $\theta = \pm\chi_m$:

$$\phi \sim \phi^{\mathrm{In}} + f_0(z) \sum_{m\in\mathcal{M}} A_m^{\pm} \mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\chi_m)}, \qquad (18)$$

where

$$A_m^{\pm} = \frac{\pi\mathrm{i}}{kL}\frac{1}{\sin\chi_m}\sum_{\mu=-\infty}^{\infty} A_{0\mu}\,\mathrm{e}^{\pm\mathrm{i}\mu\chi_m} \qquad (19)$$

and $A_{0\mu}$ are the coefficients of the line-array solution (in particular, this satisfies (10) and (11) with $q_1 = k\cos\chi$) in the standard cylindrical eigenfunction expansion of outgoing waves (see Peter *et al.*, 2006, for details). It is implicit in the above that $\sin\chi_m \neq 0$ for all m.

Thus, for given k, L and χ, the far-field scattering characteristics of a line array are completely described by the reflection and transmission matrices $\mathfrak{r}, \mathfrak{t} \in \mathbb{C}^{\#\mathcal{M}\times\#\mathcal{M}}$, in which the coefficients A_m^- and $\delta_{m0} + A_0^+$, respectively, are saved, calculated for each incident angle χ_m.

4.2 Bloch waves and Bloch transmission

Having established the solution of a periodic line array, we now consider the case of a doubly periodic array. In terms of the notation from the previous section, we assume that, for given k, L, and χ, a single line array with reflection and transmission matrices $\mathfrak{r}^{\pm}, \mathfrak{t}^{\pm} \in \mathbb{C}^{\#\mathcal{M}\times\#\mathcal{M}}$ as defined above is repeated periodically along the y-axis with fixed period W. In order to satisfy the rest of the periodicity condition (9), we need to enforce (12) and (13).

It turns out (see Peter & Meylan, 2009*a*, for details) that possible values of $\mathrm{e}^{\mathrm{i}q_2W}$ have to be eigenvalues of the matrix

$$\begin{bmatrix} Q\mathfrak{t}^+ & Q\mathfrak{r}^- \\ -(\mathfrak{t}^-)^{-1}\mathfrak{r}^+Q\mathfrak{t}^+ & -(\mathfrak{t}^-)^{-1}\mathfrak{r}^+Q\mathfrak{r}^- + (Q\mathfrak{t}^-)^{-1} \end{bmatrix},$$

where Q is a diagonal matrix with diagonal elements $\mathrm{e}^{\mathrm{i}k\sin\chi_m W}$.

In order to be able to use the method to determine q_2 once k (or ω) and q_1 have been specified, it is important to realise that $q_1 = k\cos\chi$. Since k and χ are the input of the line-array method, χ has to be chosen such that $q_1 = k\cos\chi$.

Furthermore, it is important to note that the method of coupling the line arrays involves a wide-spacing approximation in the y-direction. The

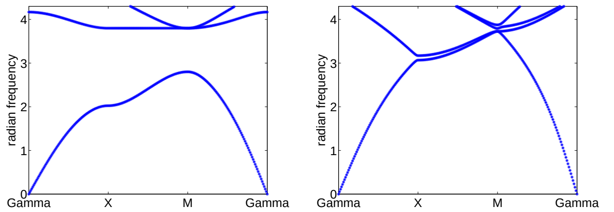

Fig. 1: Band structure of a lattice of bottom-mounted circular cylinders of radius 1.35 m (left) and 0.5 m (right), where $L = 3$ m and $d = 1.8$ m, calculated by the method of McIver (2000).

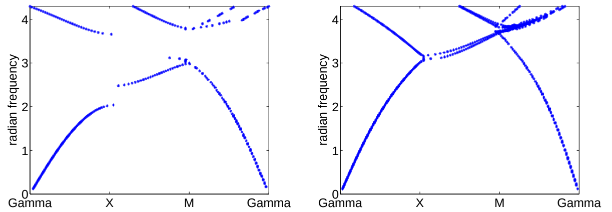

Fig. 2: Same as fig. 1 but calculated with the method presented here.

method could be extended to avoid this approximation by also including evanescent waves in (18) and this is an obvious starting point for increasing the range of applicability of the method even further.

5 Simulation results

We present some preliminary results of the band structure of square lattices (i.e. $L = W$) of bottom-mounted circular cylinders. Owing to the symmetry of the periodicity cell (and the scatterer), it suffices to compute the bands for the boundary of the first irreducible Brillouin zone, which is given by the triangle with corners $\Gamma = (0, 0)$, $X = (0, \pi/L)$ and $M = (\pi/L, \pi/L)$ in $\mathbf{q}$-space.

It is typically the case that the maximum of the frequency band occurs on a boundary of the irreducible Brillouin zone although it is very important to keep in mind this does not have to be the case in general.

The band diagram in fig. 1 (left) (where the parameters are rescaled versions of those found by Hu *et al.* (2004)) exhibits a band gap between about radian frequencies of 2.8 and 3.8, i.e. at these frequencies no waves can travel through the lattice in any direction without loss of amplitude. As the maximum of the first band lies at the M-point $(\pi/3, \pi/3)$, such an array exhibits negative refractive index (as explained in Hu *et al.*, 2004). The band diagram for the same situation but with smaller cylinders is shown in fig. 1 (right). No band gap is observed here for this frequency range.

The plots in figure 1 were obtained using the method of McIver (2000) as outlined in §3. The computation for the same parameters but using the modified method of Peter & Meylan (2009*a*) as discussed in §4 is shown in fig. 2 and it can be seen that the results agree well. For the larger cylinders (left plots) some discrepancies can be observed and these are very likely due to the wide-spacing approximation used in the method of §4. It is a topic of current investigation to avoid the wide-spacing approximation and also to run computations for other types of bodies (where truncated cylinders and circular docks are natural starting points as generalisations of the bottom-mounted cylinders).

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