Semi-analytical approach in Generalized Wagner Model

by Alexander A. Korobkin
School of Mathematics, University of East Anglia, Norwich, UK

A numerical method to solve the problem of symmetric rigid contour entering water at a given speed is presented. The method is based upon the so-called Generalized Wagner Model (GWM). This model was introduced by Zhao, Faltinsen and Aarsnes (1996). Within this model the body boundary condition is imposed on the actual position of the entering surface, the free-surface boundary conditions are linearized and imposed on the pile-up height, which should be determined as part of the solution. The hydrodynamic pressure is given by the non-linear Bernoulli equation. The hydrodynamic pressures which are less than the atmospheric pressure are disregarded. The model was investigated by Mei, Liu and Yue (1999) using conformal mapping technique.

The novelty and practical importance of the approach described below is due to special accurate treatment of the flows and the pressures close to the contact points between the entering body and water free surface. This special treatment is required for reliable prediction of the hydrodynamic pressure along the wetted part of the contour during its impact onto the water surface and the following entry. Recent studies by Malleron and Scolan (2008) showed that the predictions of the hydrodynamic forces acting on entering bodies by GWM are not as good as those by MLM (see Korobkin, 2004), when we compare the theoretical forces with the measured ones. This result is not logical because formally GWM is superior with respect to MLM. However, in both the Original Wagner Model (OWM) and in MLM the integrations of the calculated pressures are performed rather accurately in contrast to the GWM, in which calculations are done numerically without account for both the flow and pressure singularities at the contact points. GWM predicts the pressures much better that MLM (compared with the fully non-linear potential solution) but, strangely, the total hydrodynamic force by MLM better fits available experimental data. A possible explanation of this strange result was given by Malenica and Korobkin (2007) who highlighted that GWM mathematically is more complex than both OWM, which allows analytical solutions and has no difficulties with numerical treatments because computers actually are not involved, and fully non-linear potential model, in which there are no contact points due to jet flows in a proximity of the body surface and the velocity field and the pressure distribution are smooth over the wetted part of the entering body. GWM by Zhao et al. (1996) just looks simple but it is not. Intensive theoretical treatment of this model is needed to increase the potential of GWM to be used in practical numerical calculations.

In the present study it is suggested to employ analytical calculations of the flow and pressure distributions much deeper than it has been done so far. We suggest to distinguish the singular components of the flow velocity and the pressure, separate them and treat them carefully. To this aim the original problem reduces to two non-linear integral equations, where one of them serves to evaluate the conformal mapping and the second one to compute the position of the contact point. Both equations are singular but their solutions can be obtained with good accuracy. We use the conformal mapping technique as in Mei et al (1999) and Malleron and Scolan (2008). Singularities of the corresponding conformal mapping were studied by Korobkin and Iafrati (2005).

The solution derived predicts accurately the hydrodynamic force similar to MLM but additionally it gives access to the pressure distribution which is not available within MLM. Calculations of the force and the pressures by the present version of the GWM are as quick as in MLM but require precalculations which depend only on the body shape but not on its motion.

1 Formulation of the entry problem within GWM

Two-dimensional problem of a rigid body entry into ideal and incompressible liquid is considered. Initially ($t = 0$) the liquid occupies the lower half-plane $y < 0$ and is at rest. A symmetric contour touches the liquid free surface $y = 0$ at a single point, which is taken as the origin of the Cartesian coordinate system Oxy. Then the contour starts to penetrate the liquid vertically. The penetration depth $h(t)$ is assumed prescribed. Position of the contour is given by the equation $y = f(x) - h(t)$, where the even function $f(x)$, $f(-x) = f(x)$, describes the contour shape. Position of the liquid free surface after the impact, $y = S(x, t)$, $S(-x, t) = S(x, t)$, $S(x, 0) = 0$, and the horizontal dimension of the contact region, $-c(t) < x < c(t)$, between the liquid and the body are unknown in advance and have to be determined as part of the solution.

Within the Generalized Wagner Model the function $H(t) = S[c(t), t]$ is introduced. The free surface boundary conditions are linearized and imposed on the line $y = H(t)$, $|x| > c(t)$. The body boundary condition is taken in its original form and is imposed on the actual position of the body surface. The flow is symmetric with respect to the y-axis and is described by the velocity potential $\varphi(x, y, t)$. We do not discuss here motivations for this approach.

The liquid flow is governed by the equations

$$\nabla^2\varphi = 0 \qquad (y < f(x) - h(t), |x| < c(t)) \cup (y < H(t), |x| > c(t)), \tag{1}$$

$$\varphi = 0 \qquad (y = H(t), |x| > c(t)), \tag{2}$$

$$S_t(x,t) = \varphi_y(x, H(t), t) \qquad (|x| > c(t)), \tag{3}$$

$$\varphi_y = f'(x)\varphi_x - \dot{h}(t) \qquad (y = f(x) - h(t), |x| < c(t)), \tag{4}$$

$$\varphi \to 0 \qquad (x^2 + y^2 \to \infty), \tag{5}$$

$$H(t) = f[c(t)] - h(t), \tag{6}$$

$$S(x,0) = 0, \quad c(0) = 0. \tag{7}$$

If the function $c(t)$ is known, then we calculate $H(t)$ by using (6) and solve the boundary-value problem (1), (2), (4) and (5). Then we integrate the kinematic condition (3) subject to the initial conditions (7), evaluate $S[c(t),t] = H(t)$ and check equation (6). If the function $c(t)$ was given correctly, equation (6) is identically satisfied at any time instant t.

Once the boundary-value problem (1) - (7) has been solved, the hydrodynamic pressure $p(x,y,t)$ in the flow region is computed by using the non-linear and unsteady Bernoulli equation

$$p(x,y,t) = -\rho[\varphi_t + \frac{1}{2}|\nabla\varphi|^2]. \tag{8}$$

Gravity and surface tension effects are not taken into account within this approach.

Note that the conditions (2) and (4) do not match each other at the contact points. As a result, the flow velocity is singular at these points and the pressure (11) tends to $-\infty$ when we approach the contact points. In GWM, only positive pressures matter.

2 Analytical study

We introduce the stream function $\psi(x,y,t)$, the complex variable $z = x + iy$, the complex potential $w(z,t) = \varphi(x,y,t) + i\psi(x,y,t)$ and conformal mapping of the lower half plane ζ, $\Im[\zeta] < 0$, onto the flow domain as $z = iH(t) + F(\zeta,t)$. Here $F(\pm 1, t) = \pm c(t)$, $\zeta = \xi + i\eta$, $F(\zeta,t) = F_R(\xi,\eta,t) + iF_I(\xi,\eta,t)$, $x = F_R(\xi,\eta,t)$, $y = F_I(\xi,\eta,t) + H(t)$, $X(\xi,t) = F_R(\xi,0,t)$, and $w(z,t) = w[iH(t) + F(\zeta,t),t] = \tilde{w}(\zeta,t)$. The complex function $\tilde{w}(\zeta,t)$ is analytic in the lower half plane $\eta < 0$ and satisfies the following boundary conditions

$$\Re[\tilde{w}(\xi - i0, t)] = 0 \quad (|\xi| > 1), \quad \Im[\tilde{w}(\xi - i0,t)] = \dot{h}(t)X(\xi,t) \quad (|\xi| < 1), \tag{9}$$

which follow from the original boundary conditions (2) and (4), respectively. The analytic function which satisfies boundary conditions (9) is given by

$$\tilde{w}(\zeta,t) = i\dot{h}(t)\Big[F(\zeta,t) - F_\infty(t)\sqrt{\zeta^2 - 1}\Big], \tag{10}$$

where a real function $F_\infty(t)$ is the coefficient in the far field asymptotics of the conformal mapping, $F(\zeta,t) \sim F_\infty(t)\zeta$ as $|\zeta| \to \infty$.

In order to evaluate the function $X(\xi,t)$, we introduce a new unknown function

$$U(\xi,c) = X_\xi(\xi,t)\sqrt{1-\xi^2}/F_\infty(c) \quad (|\xi| < 1), \tag{11}$$

which satisfies the singular integral equation

$$f'[X(\xi,c)]U(\xi,c) - \frac{1}{\pi}\int_{-1}^{1}\frac{U(\tau,c)\mathrm{d}\tau}{\tau - \xi} = \xi \qquad (|\xi| < 1). \tag{12}$$

Note that we consider from now on the variable c as a new time-like independent variable with time t being an unknown function of c, $t = T(c)$. The unique solution of this equation is specified by the additional condition

$$\int_{-1}^{1}\frac{U(\tau,t)\mathrm{d}\tau}{1-\tau} = \pi. \tag{13}$$

Once equation (12) has been solved, we compute

$$F_\infty(c) = c\Big[\int_0^1 \frac{U(\tau,c)\mathrm{d}\tau}{\sqrt{1-\tau^2}}\Big]^{-1}, \quad X_\xi(\xi,c) = \frac{F_\infty(c)U(\xi,c)}{\sqrt{1-\xi^2}}, \quad X(\xi,c) = F_\infty(c)\int_0^\xi \frac{U(\xi,c)\mathrm{d}\xi}{\sqrt{1-\xi^2}}. \tag{14}$$

The integral equation (12) is solved by iterations for a given value of c. As a first guess we take $X^{(0)}(\xi,c) = c\xi$ and compute the solution $U(\xi,c)$. Then we calculate $F_\infty(c)$ and next approximation of $X(\xi,c)$ by (14). The obtained function $X^{(1)}(\xi,c)$ is substituted in (12) and this equation is solved again. The function $U(\xi,c)$ is sought in the form

$$U(\xi,c) = (1-\xi)^{k(c)}\ S(\xi,c), \quad k = \frac{1}{2} - \frac{\gamma}{\pi}, \tag{15}$$

where γ is the deadrise angle, $\tan\gamma(c) = f'(c)$, $0 < \gamma < \frac{\pi}{2}$, and $S(\xi,c)$ is a new unknown smooth function which has a certain value $S(1,c)$ at $\xi = 1$.

The coordinate of the contact point $c(t)$ is very important in the Wagner approaches. Within GWM this function is determined by the Wagner condition (6), where $H(t)$ is calculated by integration of the kinematic condition (3). The Wagner condition reads

$$\int_0^t \frac{\partial\varphi}{\partial y}(c(t), H(\tau), \tau)\mathrm{d}\tau = f[c(t)] - h(t), \tag{16}$$

where the vertical velocity on the free surface, $x > c(t)$, is given by the formula

$$\frac{\partial\varphi}{\partial y}(x, H(t), t) = \dot{h}(t)\Big[-1 + \frac{F_\infty(t)\xi}{X'(\xi,t)\sqrt{\xi^2-1}}\Big]. \tag{17}$$

A prime stands for the derivative with respect to ξ.

We substitute the vertical velocity (17) into the Wagner equation (16). Taking into account that $\tau \le t$, $c(\tau) \le c(t)$ and $c(t) = X(1,t)$, introducing new time-like variable c instead of t (now $t = T(c)$), new integration variable c_0 such that $\tau = T(c_0)$ and new function $\xi(c,c_0)$ defined by the equation $c = X[\xi(c,c_0), T(c_0)]$, we can rewrite equation (16) as

$$f(c) = \int_0^c N(c_0)\frac{F_\infty[T(c_0)]\xi(c,c_0)\mathrm{d}c_0}{X_\xi[\xi(c,c_0), T(c_0)]\sqrt{\xi^2(c,c_0)-1}}, \tag{18}$$

where

$$N[c(t)] = \frac{\dot{h}(t)}{\dot{c}(t)}. \tag{19}$$

The integrand in (18) is singular because $\xi(c,c_0) \to 1$ as $c_0 \to c$ (see Korobkin and Iafrati, 2005). Equation (19) serves to compute the derivative $\dot{c}(t)$ and the wetting correction function $c(t)$ by time integration of this derivative for a given velocity of penetration $\dot{h}(t)$. Equation (18) is a singular integral equation with respect to the unknown function $N(c)$. This equation can be written in the form

$$f(c) = \int_0^c N(c_0)\frac{m(c,c_0)\mathrm{d}c_0}{(1-c_0^2/c^2)^{æ(c)}}, \quad æ(c) = \frac{\pi/2 - \gamma(c)}{\pi - \gamma(c)} \tag{20}$$

where the bounded function $m(c,c_0)$ is calculated by integration of an auxiliary ODE once the conformal mapping $F(\zeta,t)$ is known. Equation (20) reduces to the classical Wagner equation when $m(c,c_0) = 1$ and $æ(c) = \frac{1}{2}$.

Substituting (10), (11) and (15) in the Bernoulli equation (8), we arrive at the following formula for the pressure distribution over the wetted part of the entering body surface in parametric form

$$p = \rho\dot{h}^2 P_v(\xi,c) + \rho\ddot{h}P_w(\xi,c), \tag{21}$$

$$P_v(\xi,c) = \frac{X_c(\xi,c)}{N(c)}\frac{\xi}{S(\xi,c)}(1-\xi)^{-k(c)} - \frac{0.5}{1+f_x^2(X)}\frac{\xi^2}{S^2(\xi,c)}(1-\xi)^{-2k(c)} + \frac{F'_\infty(c)}{N(c)}\sqrt{1-\xi^2} + \frac{1}{2} - \frac{f_x(c)}{N(c)}, \tag{22}$$

$$P_w(\xi,c) = f(X) - f(c) + F_\infty(c)\sqrt{1-\xi^2}. \tag{23}$$

It is important that the functions $P_v(\xi,c)$ and $P_w(\xi,c)$ do not depend on the body motion but on the body shape only. The derivatives $X_c(\xi,c)$ and $F'_\infty(c)$ in (22) are calculated numerically. Note that we calculate the hydrodynamic force by integrating positive hydrodynamic pressures only. This is a complicated procedure which requires the solution of the equation $p(\xi,c) = 0$.

If we integrate the pressure just along the contact region without distinguishing positive and negative pressures as it was suggested by Mei, Liu and Yue (1999), then the calculation of the total force is rather straightforward. We obtain

$$F_{mly}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{h}M_a(c)\right] + \rho\dot{h}^2(t)E(c), \tag{24}$$

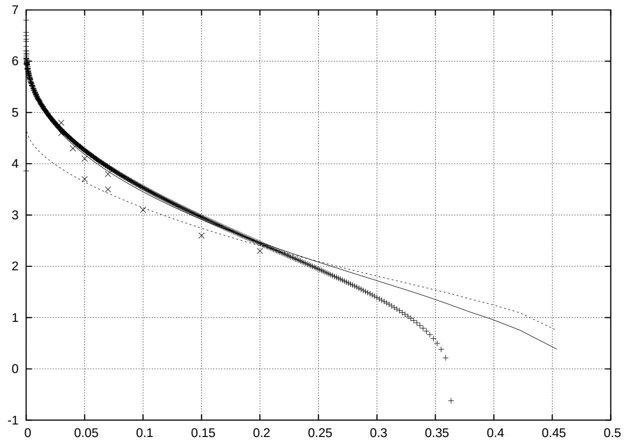
where $M_a(c)$ is the added mass of the floating body with draft $h + H$ (see Korobkin and Iafrati, 2005) and $E(c)$ is given as

$$E(c) = c - F_\infty(c)\int_0^1 \frac{\xi^2 \mathrm{d}\xi}{S(\xi,c)[1+f_x^2(X)](1-\xi)^k\sqrt{1-\xi^2}}$$

$$M_a(c) = \rho[2F_\infty^2(c)D_1(c) - S_w(c)], \quad D_1(c) = \int_0^1 U(\xi,c)\mathrm{d}\xi,$$

and S_w is the area of the submerged part of the body.

The non-dimensional force $F'(t') = F/(\rho V^2 R)$ acting on the circular cylinder of radius R entering water at constant speed V is shown in the figure below. Horizontal axis is for non-dimensional time $t' = tV/R$. Solid line is for positive pressure integration, dashed line shows the force predicted by the formula (24), crosses are for experimental results by Armand and Cointe, and pluses are for MLM.



It is seen that the forces by GWM and MLM are very close to each other for $0 < t' < 0.2$. The formula (24) underpredicts the force on this interval. However, for $0.2 < t' < 0.45$ the prediction by GWM is rather close to that by (24) but MLM underpredicts the force on this interval.

This study was carried out within the project "Generalized Wagner Approach for Symmetric Bodies" sponsored by Seoul National University (coordinator: Prof. Yonghwan Kim). Supports from FP7 projects TULCS (coordinator: Dr. Sime Malenica) and SMAES (coordinator: Dr. James Campbell) for this presentation and motivations for further development of the approach are acknowledged.

3 References

Korobkin, A.A., 2004, Analytical models of water impact. European Journal of Applied Mathematics. Vol. 15, pp. 821-838.

Korobkin A.A., Iafrati A., 2005, Hydrodynamic loads during initial stage of floating body impact. J. Fluids and Structures. Vol. 21. No. 4, pp. 413-427.

Malenica S., and Korobkin A. A., 2007, Some Aspects of Slamming Calculations in Seakeeping. In: Proceedings of 9th International Conference on Numerical Ship Hydrodynamics, University of Michigan.

Malleron, N., Scolan, Y.M., 2008, Generalized Wagner model for 2D symmetric and elastic bodies, IWWWFB23 13-16 April 2008 Jeju, Korea.

Mei X., Liu Y., Yue D.K.P, 1999, On the water impact of general two-dimensional sections. Applied Ocean Research. Vol. 21, pp. 1-15.

Zhao R., Faltinsen O., Aarsnes J., 1996, Water entry of arbitrary two-dimensional sections with and without flow separation. In: Proceedings of 21th Symposium on Naval Hydrodynamics.