Transmission of water waves through apertures in a pair of thin vertical barriers

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1. Introduction

Tuck(1971) employed a method based on matched asymptotic expansion to find approximately the transmission coefficient for the two-dimensional problem of water wave transmission through a small aperture in an impermeable vertical barrier. Porter(1972) used a reduction procedure and also an integral equation formulation based on Cauchy type integral equation to solve this problem for an aperture of any size and obtained the transmission and reflection coefficients in closed forms in terms of computable integrals. Instead of a single barrier one can consider two thin vertical impermeable barriers having apertures of the same size and at same depth below the mean free surface. The corresponding wave transmission problem has been solved here explicitly by reducing it to solving Able integral equations. The transmission and reflection coefficients are obtained in terms of expressions involving computable integrals. The transmission coefficient ($|T|$) is depicted graphically against the wave number in two figures for different values of the ratio of width of the apertures to the depth (h) of its mid point and the ratio of separation length and h. When the two barriers are very close to each other, the curves for transmission and reflection ($|R|$) coefficients coincide with those given in Porter(1972).

2. Mathematical formulation

Assuming linearised theory and irrotational motion, the problem of our interest is to solve for $\phi(x,y)$ satisfying

$$\nabla^2\phi = 0 \quad y \geq 0, \tag{2.1}$$

$$K\phi + \phi_y = 0 \quad \text{on} \;\; y = 0, \tag{2.2}$$

$$\phi_x = 0 \quad \text{on} \;\; x = \pm a, \; y \in (0,b) \cup (c,\infty), \tag{2.3}$$

$$r^{1/2}\nabla\phi = O(1) \quad \text{as} \;\; r = \{(x \pm a)^2 + (y-d)^2\}^{1/2} \to 0, (d = b, c) \tag{2.4}$$

$$\nabla\phi \to 0 \;\; \text{as} \;\; y \to \infty, \tag{2.5}$$

$$\phi(x,y) \sim \begin{cases} T\phi_0(x,y) & \text{as } x \to \infty, \\ \phi_0(x,y) + R\phi_0(-x,y) & \text{as } x \to -\infty, \end{cases} \tag{2.6}$$

where $\mathrm{Re}\{\phi(x,y)e^{-i\sigma t}\}$ denotes the velocity potential describing the motion in the fluid region ($e^{-i\sigma t}$ being dropped throughout), T and R denote the unknown transmission and reflection coefficients, $\phi_0(x,y) = e^{-Ky+iKx}$ is the potential of the wave train incident from the direction of $x = -\infty, x = \pm a, (y \in (0,b) \cup (c,\infty))$ denote the positions of the two barriers below the mean free surface $y = 0$, y-axis being taken vertically downwards into the fluid region.

3. Solution

Using Havelock's expansion of water wave potential, $\phi(x,y)$ has the representation

$$\phi(x,y) = \begin{cases} e^{-Ky}(e^{iKx} + Re^{-iKx}) + \dfrac{2}{\pi}\displaystyle\int_0^\infty A(\xi)L(\xi,y)e^{\xi x}d\xi, & x < -a, \\ e^{-Ky}(\alpha e^{iKx} + \beta e^{-iKx}) + \dfrac{2}{\pi}\displaystyle\int_0^\infty \{B(\xi)e^{\xi x} + C(\xi)e^{-\xi x}\}L(\xi,y)d\xi, & -a < x < a, \\ Te^{-Ky+iKx} + \dfrac{2}{\pi}\displaystyle\int_0^\infty D(\xi)L(\xi,y)e^{-\xi x}d\xi, & x > a, \end{cases} \tag{3.1}$$

where $L(\xi,y) = \xi\cos\xi y - K\sin\xi y$, α,β are unknown constants, $A(\xi), B(\xi), C(\xi)$ and $D(\xi)$ are unknown functions such that integrals in (2.7) are convergent. The constants α,β and R, T can be shown to satisfy the relations (cf. De *et.al.*(2009,2010))

$$e^{-iKa} - Re^{iKa} = \alpha e^{-iKa} - \beta e^{iKa}, Te^{iKa} = \alpha e^{iKa} - \beta e^{-iKa}. \tag{3.2}$$

An appropriate modification of the mathematical analysis given in De *et.al.*(2009,2010) leads to

$$f(b) = 0, \int_b^c \frac{f'(y)}{(c^2-y^2)^{1/2}}dy = 0. \tag{3.3}$$

and

$$F(b) = 0, \int_b^c \frac{F'(y)}{(c^2-y^2)^{1/2}}dy = 0. \tag{3.4}$$

where the functions $f(y)$ and $F(y)$ are given by

$$f(y) = f_1(y) + f_2(y), \quad F(y) = f_1(y) - f_2(y), \tag{3.5}$$

$$\begin{aligned}(f_1(y), f_2(y)) = {} & \frac{i}{2}\left(\alpha e^{iKa} - \beta e^{-iKa}\right)\left\{\int_o^b \sinh Kt\mathcal{K}_1(t,y)dt - \frac{i}{2}\int_c^\infty e^{-Kt}\mathcal{K}_1(t,y)dt\right\} \\ & + \frac{i}{2}\left(\beta e^{iKa} - \alpha e^{-iKa}\right)\left\{\int_o^b \sinh Kt\mathcal{K}_2(t,y)dt - \frac{i}{2}\int_c^\infty e^{-Kt}\mathcal{K}_2(t,y)dt\right\} \\ & + \frac{\pi}{4K}e^{-Ky}\left(\beta e^{-iKa}, (\beta - R)e^{iKa}\right) + (C_2, C_1)e^{Ky}, b < y < c,\end{aligned} \tag{3.6}$$

where C_1, C_2 are some unknown constants. The six relations (3.2) to (3.4) are sufficient to determine the six unknown constants $R, T, \alpha, \beta, C_1, C_2$. In the next section we find the transmission and reflection coefficients.

4. Reflection and transmission coefficients

Substituting the appropriate expressions for $f(y)$ and $F(y)$ in (3.3) and (3.4), carrying out the necessary integrations, and substituting for α, β in terms of T, R from (3.2), we ultimately obtain two equations involving R and T. These give T and R as

$$(T,R) = \frac{1}{2}\left[\frac{\pi I + 4e^{-iKa}U\sin Ka}{\pi I + 4e^{iKa}U\sin Ka} \mp \frac{4ie^{-iKa}V\cos Ka - \pi I}{4ie^{iKa}V\cos Ka + \pi I}\right] \tag{4.1}$$

with $U = U_1 e^{Kb} - U_2 J_1$, $I = J_2 e^{Kb} + J_1 e^{-Kb}$, $V = V_1 e^{Kb} - V_2 J_1$ where

$$(U_1, V_1) = \int_0^\infty \left[\frac{M(\xi, K)(e^{2\xi a} \pm 1)}{(\xi^2 + K^2) \sinh 2\xi a} \int_b^c \frac{\cos \xi x}{(c^2 - x^2)^{1/2}} dx \right] d\xi, \tag{4.2}$$

$$(U_2, V_2) = \int_0^\infty \left[\frac{M(\xi, K)(e^{2\xi a} \pm 1) K \sin \xi c}{(\xi^2 + K^2)\xi \sinh 2\xi a} \right] d\xi, (J_1, J_2) = \int_b^c \frac{(e^{Kx}, e^{-Kx})}{(c^2 - x^2)^{1/2}} dx \tag{4.3}$$

$$M(\xi, K) = (K \cosh Kb \sin \xi b - \xi \sinh Kb \cos \xi b) - \frac{1}{2} e^{-Kc} (\xi \cos \xi c + K \sin \xi c).$$

It is verified that T and R satisfy the energy identity $|T|^2 + |R|^2 = 1$. Also α, β can be found from equation (3.2) using these expressions for T and R.

5. Discussion

5.1 *Approximation of T, R for small separation length*

It is possible to derive the results for a single thin semi-infinite vertical plane barrier with an aperture from (4.1) by making $a \to 0$. We find that

$$U_j \sin Ka \to U_{j0}, \quad V_j \to V_{j0} \quad (j = 1, 2) \quad \text{as} \quad a \to 0$$

where

$$(U_{10}, U_{20}) = K \int_0^\infty \left[\frac{M(\xi, K)}{(\xi^2 + K^2)\xi} \left(\int_b^c \frac{\cos \xi x}{(c^2 - x^2)^{1/2}} dx, \frac{K \sin \xi b}{\xi} \right) \right] d\xi, (V_{10}, V_{20}) = \frac{1}{K}(U_{10}, U_{20}).$$

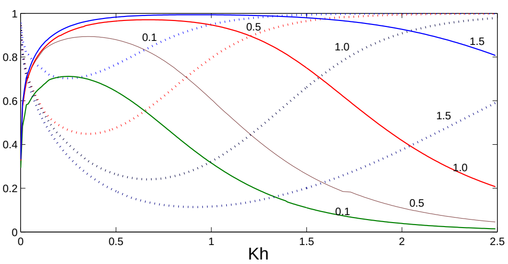

Figure 1: $|T_1|$ (-) and $|R_1|(\cdots)$ against Kh for different values of μ

Using these results in (4.1), we find that as $a \to 0$

$$T \to T_1 \equiv \frac{\pi I}{\pi I + 4iW} \quad \text{and} \quad R \to R_1 \equiv \frac{4iW}{\pi I + 4iW}, \tag{5.1}$$

where $W = V_{10} e^{Kc} - V_{20} J_1$, J_1 being given in (4.4). If we write $\mu = \frac{c-b}{h}$ where $h = \frac{b+c}{2}$ then $b = h(1 - \frac{\mu}{2}), c = h(1 + \frac{\mu}{2})$ so that $0 < \mu < 2$. In figure 1, $|T_1|$ and $|R_1|$ are depicted against Kh for $\mu = 0.1, 0.5, 1.0$ and 1.5. The curves for $|T_1|$ and $|R_1|$ in figure 1 coincide with the corresponding curves given in Porter(1972) who employed different methods to study the problem of a single barrier with a submerged gap. This shows the correctness of the mathematical analysis employed here.

5.2 *Numerical results*

Because of the energy identity it is sufficient to consider the transmission coefficient $|T|$ only. $|T|$ is depicted graphically against Kh in figures 2, 3 for $a/h = 5, 10(\mu = 1.0)$. The curve for $|T|$ becomes more oscillatory as separation ratio increases. The oscillatory nature is attributed to multiple reflections of the incident wave train by the two barriers. Also complete transmission occurs (for which $|T| = 1$) at certain discrete frequencies. The number of these frequencies increases with the increase in the separation ratio (a/h) of the two barriers.

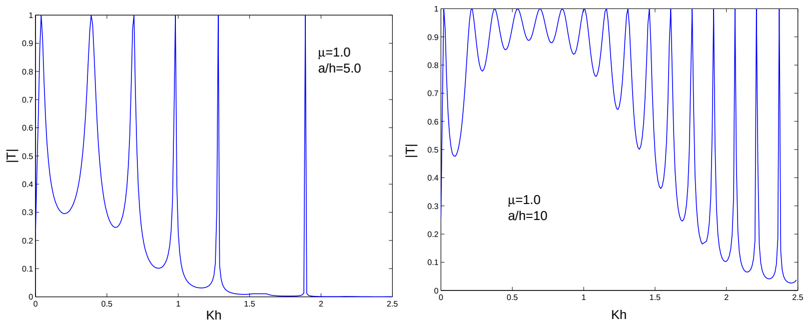

Figure 2: $|T|$ against Kh

6. Conclusion

A method based on the solution of Abel integral equation is employed here to obtain an explicit solution to the problem of water wave transmission involving two vertical barriers with apertures at the same level in deep water. This problem appears to have not been investigated earlier in the literature.

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References

1. E.O. Tuck, Transmission of water waves through small apertures, *J. Fluid Mech.* 49 (1971) 65 - 74.

2. D. Porter, The transmission of surface waves through a gap in a vertical barrier. *Proc. Camb. Phil. Soc.* 71 (1972) 411 - 421.

3. S.De, B.N. Mandal and A. Chakrabarti, Water wave scattering by two submerged plane vertical barriers - Abel integral equations approach. *J. Engng. Math.* 65 (2009) 75-87.

4. S.De, B.N. Mandal and A. Chakrabarti, Use of Abel integral equations in water wave scattering by two surface-piercing barriers. *Wave Motion* 47 (2010) 279-288.