On the non-linear evolution of directionally spread wave-groups

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1 Introduction

To calculate the loading and interactions of large waves with maritime structures it is important to understand the non-linear changes which occur as a large wave-group forms. Much work has gone into looking at these non-linear changes over the last twenty years, much of it to investigate whether these may be the cause of 'freak' or 'rogue' waves. The consensus is that in uni-directional seas, non-linear changes (the Benjamin-Feir instability) cause a significant increase in the size of the wave relative to those which would form under linear evolution Janssen (2003). However, for realistic, directionally spread seas there is little or no extra elevation due to this instability (see for instance Gramstad & Trulsen (2007)).

Whilst non-linear dynamics do not appear to cause abnormally large waves, they do cause significant changes to the shape of large waves relative to those found under linear evolution, which will cause significant differences to the way in which the largest waves interact with fixed and floating structures. The dominant change is that there is a contraction of the wave-group in the mean wave direction and an increase in the width of the wave in the lateral direction causing a significant local reduction in the directional spreading under the crest. These changes have been found in physical and numerical experiments by Johannessen & Swan (2001); Gibbs & Taylor (2005) and have been observed in the open ocean by Krogstad *et al.* (2006). The changes to the group shape are very significant, at least for isolated wave-groups as is shown in Figure 1. This also shows a comparison between the changes to the shape of a wave-group modelled using the full potential flow water-wave equations and those found using a simpler cubic non-linear Schrödinger equation (NLSE) model (see section 2). It can be seen that whilst the NLSE results do not capture the detail of the changes in the group shape, it does capture the overall changes to the group which we are seeking to investigate in this work. We use the NLSE to derive analytical relationship between the shape of a wave-group under linear evolution with that under non-linear evolution. This work follows on from that of Adcock & Taylor (2009*b*) where a similar approach was taken for describing the changes to uni-directional wave-groups.

2 Governing equations

The hyperbolic 2-D NLSE (equation 1) provides the simplest non-linear model for the evolution of water waves on deep water.

$$i\frac{\partial u}{\partial t} = \frac{\omega_0}{8k_0^2}\frac{\partial^2 u}{\partial x^2} - \frac{\omega_0}{4k_0^2}\frac{\partial^2 u}{\partial y^2} + \frac{\omega_0 k_0^2}{2}\left|u\right|^2 u, \tag{1}$$

where u is the complex wave envelope, and ω_0 and k_0 are the frequency and wavenumber of the carrier wave which is traveling in the x direction. This may be non-dimensionalised using the transformations $T = \omega_0 t$, $X = 2\sqrt{2}k_0 x$, $Y = 2k_0 y$ and $U = \left(k_o/\sqrt{2}\right) u$ to give

$$i\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial Y^2} + \left|U\right|^2 U. \tag{2}$$

This paper makes use of the conservation laws of the NLSE. The 2D version has a finite number of conserved quantities of which only two non-trivial and useful ones are known (Sulem & Sulem, 1999). These are

$$I2 = \int_{Y=-\infty}^{Y=\infty}\int_{X=-\infty}^{X=\infty} \left|U\right|^2 dX dY, \tag{3}$$

$$I4 = \int_{Y=-\infty}^{Y=\infty}\int_{X=-\infty}^{X=\infty} \left|U_X\right|^2 - \left|U_Y\right|^2 - \frac{1}{2}\left|U\right|^4 dX dY. \tag{4}$$

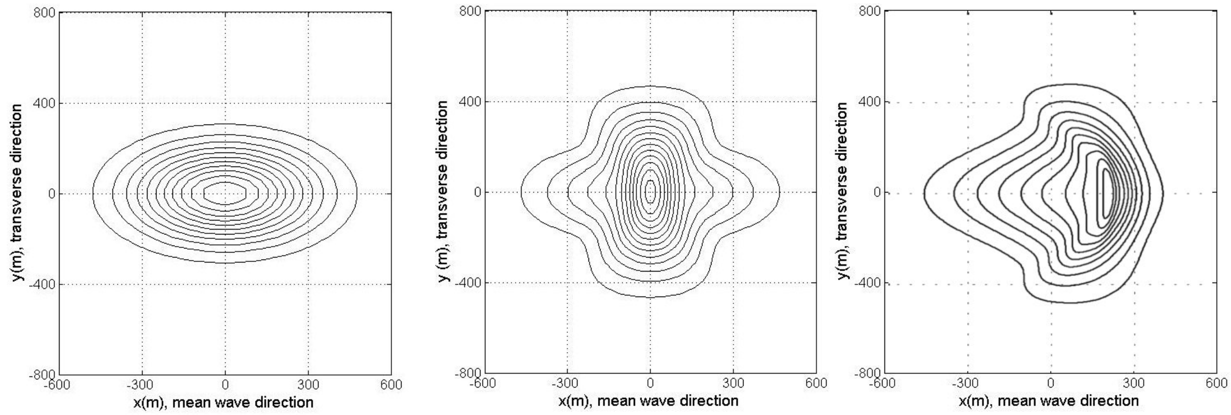

Figure 1: Wave-groups at focus for which were started 20 periods before linear focus and would have an amplitude of 10.7m under linear evolution. Left shows linear evolution; centre shows evolution using the NLSE; right shows fully non-linear evolution.

Adcock (2009) attempted to extend the next useful conserved quantity of the uni-directional NLSE to the 2D form. However, it was found that this was not exactly conserved. Thus, no other, non trivial, conserved quantity is known.

3 Analytical results

A solution to the linear part of equation 2 for a focusing Gaussian group can be derived from Kinsman (1965). This is given by equation 5. A Gaussian may be used as an approximation to a more realistic sea-state directional spectrum if only the spectral peak is considered. This linear solution is

$$U = \frac{A}{\sqrt{1+2iS_X^2T}\sqrt{1+2iS_Y^2T}}\mathrm{Exp}\left[-\frac{1}{2}\left(\frac{S_X^2X^2}{1+4S_X^4X^2T^2}+\frac{S_Y^2Y^2}{1+4S_Y^4T^2}\right)+\frac{i}{2}\left(\frac{S_Y^2Y^2}{1+4S_Y^4T^2}-\frac{S_X^2X^2}{1+4S_X^4T^2}\right)\right]. \tag{5}$$

In this analysis, the parameters A, S_X and S_Y are the amplitude and bandwidths of the directional wavenumber spectrum, defined as the amplitude and bandwidth of a group if it was perfectly focused.

We now assume that the complex wave envelope remains Gaussian in form but that the parameters A, S_X and S_Y vary slowly with a non-linear timescale τ. To determine how these vary, we substitute equation 5 into equations 3 and 4. This gives two equations with three unknowns.

$$I2 = \frac{A^2}{S_XS_Y}. \tag{6}$$

$$I4 = \frac{\pi A^2\left(S_X^2-S_Y^2\right)}{2S_XS_Y} - \frac{\pi A^4\left(S_XS_Y\right)^{-1}}{4\left(1+4S_X^2\tau^2\right)^{\frac{1}{2}}\left(1+4S_Y^2\tau^2\right)^{\frac{1}{2}}}. \tag{7}$$

We may now substitute in to these equations $\tau = 0$ (focus) and $\tau = \infty$ (fully dispersed). As we have only two equations and three unknowns we must make a further assumption to find the relationship between the parameters at focus compared to those for a completely dispersed group. The fully non-linear simulations of Gibbs & Taylor (2005), and subsequent numerical solutions of the NLSE, found that A remained virtually unchanged even for highly non-linear wave-groups. Thus we make the assumption that A remains constant in time and may thus solve to relate the shape of wave group at focus relative to that at infinity (which is also the shape which a wave-group would have at focus under linear evolution).

$$\left(\frac{S_{X\infty}}{S_{Xf}}\right)^2 = \frac{1}{4}\left(2-\left(\frac{A}{S_{Xf}}\right)^2-2\left(\frac{S_{Yf}}{S_{Xf}}\right)^2+\sqrt{\left(2-\left(\frac{A}{S_{Xf}}\right)^2-2\left(\frac{S_{Yf}}{S_{Xf}}\right)^2\right)^2+16\left(\frac{S_{Yf}}{S_{Xf}}\right)^2}\right) \quad (8)$$

$$\left(\frac{S_{Y\infty}}{S_{Yf}}\right)^2 = \frac{1}{4}\left(2+\left(\frac{A}{S_{Yf}}\right)^2-2\left(\frac{S_{Xf}}{S_{Yf}}\right)^2+\sqrt{\left(2+\left(\frac{A}{S_{Yf}}\right)^2-2\left(\frac{S_{Yf}}{S_{Xf}}\right)^2\right)^2+16\left(\frac{S_{Xf}}{S_{Yf}}\right)^2}\right) \quad (9)$$

$$\left(\frac{S_{Xf}}{S_{X\infty}}\right)^2 = \frac{1}{4}\left(2+\left(\frac{A}{S_{X\infty}}\right)^2-2\left(\frac{S_{Y\infty}}{S_{X\infty}}\right)^2+\sqrt{\left(-2-\left(\frac{A}{S_{X\infty}}\right)^2+2\left(\frac{S_{Y\infty}}{S_{X\infty}}\right)^2\right)^2+16\left(\frac{S_{Y\infty}}{S_{X\infty}}\right)^2}\right) \quad (10)$$

$$\left(\frac{S_{Yf}}{S_{Y\infty}}\right)^2 = \frac{1}{4}\left(2-\left(\frac{A}{S_{Y\infty}}\right)^2-2\left(\frac{S_{X\infty}}{S_{Y\infty}}\right)^2+\sqrt{\left(2-\left(\frac{A}{S_{Y\infty}}\right)^2-2\left(\frac{S_{X\infty}}{S_{Y\infty}}\right)^2\right)^2+16\left(\frac{S_{X\infty}}{S_{Y\infty}}\right)^2}\right). \quad (11)$$

These show the general trend, observed in the introduction, of the group contracting in the mean wave direction and expanding in the lateral direction as it focusses. We note that there is no limit to the non-linearity of a group at focus, even in the uni-directional limit. This is in contrast to the uni-directional result derived by Adcock & Taylor (2009*b*) who found $A/S_{Xf} < 2^{1/4}$ for groups starting from fully dispersed initial conditions. This inconsistency is due our assumption that A remains constant being invalid in the uni-directional limit.

4 Comparison to numerical results

The analytical results may be compared to the results of numerical simulations. We use a fourth order Runge-Kutta scheme in time and a pseudo-spectral scheme in space to solve the NLSE. We start with a wave-group at focus and allow it to disperse. When the spectrum stops changing and the evolution of the group becomes essentially linear we fit a Gaussian to the two-dimensional spectrum. The results of this are shown in Figure 2. These show excellent agreement with the analytical results above.

Gibbs & Taylor (2005) carried out fully non-linear simulations of focusing wave-groups in deep water, using the scheme developed by Bateman *et al.* (2001). The simulations used a Gaussian wave-packet as the initial conditions with a spectrum based on a JONSWAP spectrum with $\gamma = 3.3$ and with a r.m.s. directional spreading of $15°$ with a peak wave-number of $k_p = 0.0279\text{m}^{-1}$ – these are taken to be representative of a winter storm in the North Sea. This gave a Gaussian group with $s_x = 0.0046\text{m}^{-1}$ and $s_y = 0.0073\text{m}^{-1}$. This when non-dimensionalised gives $S_{X\infty}/S_{Y\infty} = 0.45$. The most non-linear case run, $ak_0 = 0.33$ equates to $A_\infty/S_{X\infty} = 4.0$. The simulations were started 20 periods before linear focus and run for a number of different amplitudes. This is sufficiently long before focus that the resulting evolution is very similar to that of a group converging from infinity. Both 'crest' and 'trough' focused runs were carried out, allowing the odd and even order harmonics to the be separated (see Adcock & Taylor (2009*a*)) and thus the effect of bound harmonics may be removed leaving only freely propagating waves. The results presented here are based on the maximum amplitude of the freely propagating waves recorded.

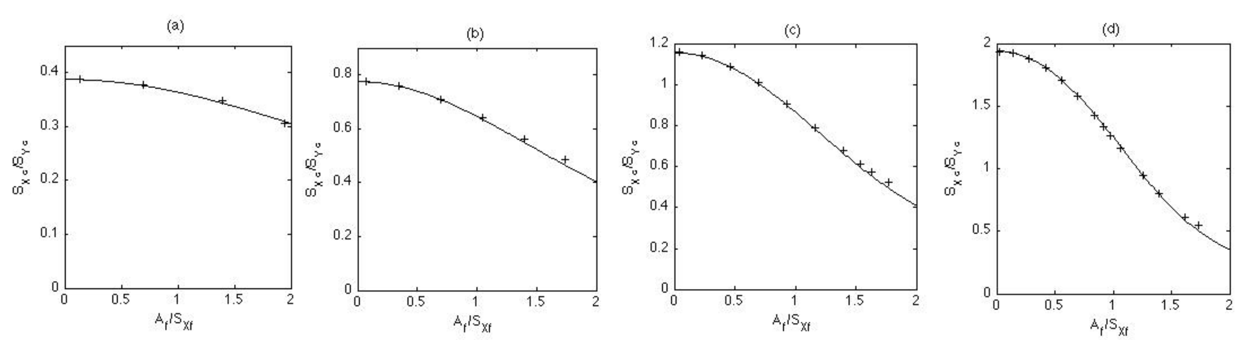

Figure 2: The group shape at infinity for various initially focused Gaussian groups. Solid line is the analytical predictions and crosses the numerical results. (a) $S_{Xf}/S_{Yf} = 0.387$, (b) $S_{Xf}/S_{Yf} = 0.775$ (c) $S_{Xf}/S_{Yf} = 1.16$, (d) $S_{Xf}/S_{Yf} = 1.94$

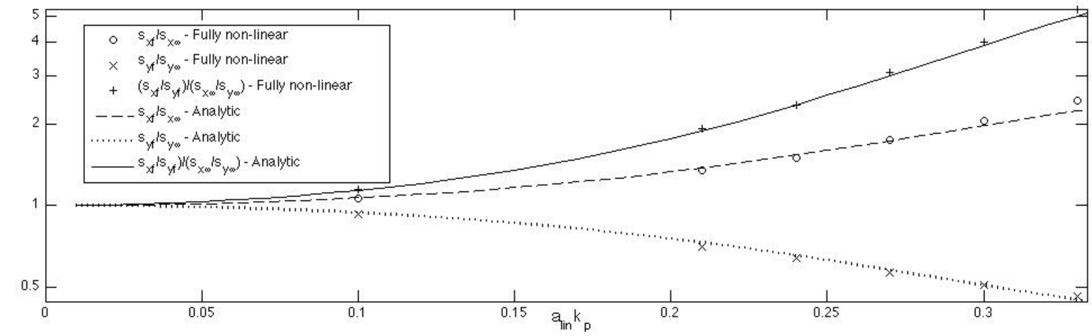

Figure 3: Comparison of predicted change to group shape with the fully non-linear results of Gibbs & Taylor (2005).

As noted in the introduction, very little extra elevation was observed when compared to linear evolution. However, there is a significant change in the group shape due to the non-linear evolution. These changes may be quantified by fitting a Gaussian to the wave-group. Note that the values presented here are different from those in Gibbs & Taylor (2005) who fitted a Gaussian only to the peak of the group whereas here we are fitting the whole group. Estimation of the s_{xf} fit was performed along the x axis. The value of s_{yf} has been inferred from conservation of energy. In figure 3 we plot the predicted group shape against that observed by Gibbs & Taylor (2005).

5 Conclusions

These results have been derived using a very simplistic approach, however, they agree surprisingly well with numerical simulations of the full water wave equations. Thus they may be used as a first approximation for describing the non-linear changes a wave-group undergoes as it focuses.

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