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A Note on Three-Dimensional Green-Naghdi Theory

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A fully nonlinear numerical model based on the Green-Naghdi equations (Demirbilek and Webster, 1992) is investigated and applied to 3-dimensional water waves. The linear analytical solution corresponding to Level I up to Level VII G-N shallow water waves has been derived. It's found that the difference of dispersive relation between Level VII G-N theory and the linear Stokes wave theory reached $O[(kh)^{17}]$, and Level VII G-N theory can predict the waves with $kh \leq 33$, but the highest derivative is only third order. An experiment conducted by Chawla and Kirby (1996) was reproduced numerically. The G-N theory presented some advantages in some details compared with the fully nonlinear Bousssinesq model. The G-N theory also can be used to simulate deep water waves. It's shown that the results of G-N theory are very close to the stream function wave theory. By making $\partial\alpha/\partial t \neq 0$, G-N theory can simulate earthquake-induced tsunami.

1. Introduction

The Green-Naghdi equations, called G-N equations for short, were originally developed in 1974 to analyze some nonlinear free-surface flows (Green et al., 1974). In G-N models the dimension of a free surface problem is reduced by one, and nonlinear boundary conditions are satisfied on the instantaneous free surface. Xu et al. (1993a, 1993b) developed the G-N theory into 3-D deep water waves. An iterative algorithm is introduced for the solution of 2-D and 3-D G-N models. The propagation of nonlinear, irregular, uni- and multi-directional waves in deep water was simulated. Xu et al. (1997) developed a wave-absorbing beach for G-N models.

Xu et al. need to construct a numerical balanced model in order to converge quickly which need special dealing. The process named "numerical balancing" may have problem in dealing with the 3-D shallow water waves. The algorithm of 3-D G-N model used here is similar with that of Boussinesq model (Wei and Kirby, 1995). This paper is organized as follows.

In section 2, the governing equations for 3-D Green-Naghdi theory are briefly derived. And the dispersion of linear G-N theory in shallow water waves is discussed. Three test cases are reported in section 3. Lastly, the conclusion is given in section 4.

2. Mathematical formulation

We introduce a three-dimensional inertial Cartesian coordinate system with the Oz axis pointed vertically upwards and the Ox axis pointed horizontally to the right. The horizontal components and vertical component of the fluid at a point are denoted by $u(x, y, z, t)$ and $v(x, y, z, t)$, and $w(x, y, z, t)$, respectively. The free surface and the bottom are defined by $z = \beta(x, y, t)$ and $z = \alpha(x, y, t)$, respectively. This paper will be concerned only with an incompressible and inviscid fluid, and the mass density of the fluid is constant. G-N theory do not introduces any small parameters. Only the following assumption was introduced:

$$u(x,y,z,t) = \sum_{n=0}^{K} u_n(x,y,t)\lambda_n(z)\,;\; v(x,y,z,t) = \sum_{n=0}^{K} v_n(x,y,t)\lambda_n(z)\,;\; w(x,y,z,t) = \sum_{n=0}^{K} w_n(x,y,t)\lambda_n(z) \quad (1)$$

in which $\lambda_n(z)$ is the base function and the coefficients (u_n, v_n, w_n) are unknown functions of (x, y, t). K is the "level" of the Green-Naghdi model. The kinematic free-surface condition and bottom condition :

$$\frac{\partial\beta}{\partial t} = \sum_{n=0}^{K} \lambda_n(\beta)\left(w_n - u_n\frac{\partial\beta}{\partial x} - v_n\frac{\partial\beta}{\partial y}\right)\;;\quad \frac{\partial\alpha}{\partial t} = \sum_{n=0}^{K} \lambda_n(\alpha)\left(w_n - u_n\frac{\partial\alpha}{\partial x} - v_n\frac{\partial\alpha}{\partial y}\right) \quad (2)$$

The continuity equation :

$$\sum_{n=0}^{K} \lambda_n(z)\left(\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y}\right) + \sum_{n=0}^{K} w_n\frac{\partial\lambda_n}{\partial z} = 0 \quad (3)$$

The conservation equation of momentum :

$$\sum_{m=0}^{K}\left[\frac{\partial u_m}{\partial t}H_{mn}+\sum_{r=0}^{K}\left(\frac{\partial u_m}{\partial x}u_r+\frac{\partial u_m}{\partial y}v_r\right)H_{mrn}+u_m\sum_{r=0}^{K}w_rH_{rn}^{m}+\mu u_mH_{mn}\right]=$$
$$\left[-\frac{\partial P_n}{\partial x}+\hat{p}\lambda_n(\beta)\frac{\partial \beta}{\partial x}-\bar{p}\lambda_n(\alpha)\frac{\partial \alpha}{\partial x}\right]/\rho \quad (4a)$$

$$\sum_{m=0}^{K}\left[\frac{\partial v_m}{\partial t}H_{mn}+\sum_{r=0}^{K}\left(\frac{\partial v_m}{\partial x}u_r+\frac{\partial v_m}{\partial y}v_r\right)H_{mrn}+v_m\sum_{r=0}^{K}w_rH_{rn}^{m}+\mu v_mH_{mn}\right]=$$
$$\left[-\frac{\partial P_n}{\partial y}+\hat{p}\lambda_n(\beta)\frac{\partial \beta}{\partial y}-\bar{p}\lambda_n(\alpha)\frac{\partial \alpha}{\partial y}\right]/\rho \quad (4b)$$

$$\sum_{m=0}^{K}\left[\frac{\partial w_m}{\partial t}H_{mn}+\sum_{r=0}^{K}\left(\frac{\partial w_m}{\partial x}u_r+\frac{\partial w_m}{\partial y}v_r\right)H_{mrn}+w_m\sum_{r=0}^{K}w_rH_{rn}^{m}+\mu w_mH_{mn}\right]=$$
$$\left[P_n'-\rho gH_n-\hat{p}\lambda_n(\beta)+\bar{p}\lambda_n(\alpha)\right]/\rho \quad (4c)$$

for $n = 1, 2, \cdot, K$. In (4), ρ is the mass density of the fluid, p the pressure, and the artificial damping coefficient μ which is a given function of (x, y) and first introduced into G-N model by Xu et al. (1997), and we have used :

$$H_n=\int_{\alpha}^{\beta}\lambda_n dz\,;\; H_{mn}=\int_{\alpha}^{\beta}\lambda_m\lambda_n dz\,;\; H_{mrn}=\int_{\alpha}^{\beta}\lambda_m\lambda_r\lambda_n dz\,;\; H_{rn}^{m}=\int_{\alpha}^{\beta}\frac{\partial\lambda_m}{dz}\lambda_r\lambda_n dz$$

$$P_n=\int_{\alpha}^{\beta}p\lambda_n\,dz\,;\; P_n'=\int_{\alpha}^{\beta}p\frac{\partial\lambda_n}{dz}\,dz$$

while $\bar{p}$ is the unknown pressure on the bottom, and $\hat{p}$ is the pressure on the free surface which should be zero if we exclude the surface tension effects. The base functions $\lambda_n(z) = z^n$ for shallow water and $\lambda_n(z) = e^{az}z^n$ for deep water.

The dispersion of Linear G-N theory in shallow water waves are illustrated on Figure 1.

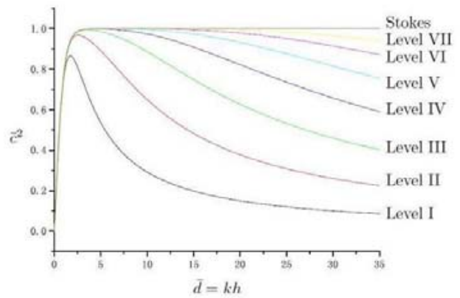

Figure 1: Dispersion of linear G-N theory

For level VII G-N theory, the approximate dispersion relations is

$$\bar{c}^2=\frac{r_1}{r_2}=\bar{d}-\frac{\bar{d}^3}{3}+\frac{2\bar{d}^5}{15}-\frac{17\bar{d}^7}{315}+\frac{62\bar{d}^9}{2835}-\frac{1382\bar{d}^{11}}{155925}+\frac{21844\bar{d}^{13}}{6081075}-\frac{929569\bar{d}^{15}}{638512875}+\frac{620626600207\bar{d}^{17}}{1051860569760000}+O(\bar{d}^{19}) \quad (5)$$

where,

$$r_1=63\bar{d}(1113079968000+165534969600\bar{d}^2+6140534400\bar{d}^4+85409280\bar{d}^6+502260\bar{d}^8+1220\bar{d}^{10}+\bar{d}^{12})$$

$$r_2=70124037984000+33803382412800\bar{d}^2+2304776073600\bar{d}^4+$$
$$50993712000\bar{d}^6+458752140\bar{d}^8+1753920\bar{d}^{10}+2583\bar{d}^{12}+\bar{d}^{14}$$

The dispersion relations of linear Stokes theory are $\bar{c}^2 = \tanh(\bar{d})$ and the Taylor series of which is

$$\bar{c}^2 = \bar{d} - \frac{\bar{d}^3}{3} + \frac{2\bar{d}^5}{15} - \frac{17\bar{d}^7}{315} + \frac{62\bar{d}^9}{2835} - \frac{1382\bar{d}^{11}}{155925} + \frac{21844\bar{d}^{13}}{6081075} - \frac{929569\bar{d}^{15}}{638512875} + \frac{6404582\bar{d}^{17}}{10854718875} + O(\bar{d}^{19}) \tag{6}$$

The difference between (5) and (6) is only $O(\bar{d}^{17})$.

3. Tests

There are three tests presented here. The first concerns a series of physical experiments for wave transformation over a circular shoal conducted by Chawla and Kirby (1996). This case is reproduced with G-N theory numerically and the results are shown on the right of Figure 2.

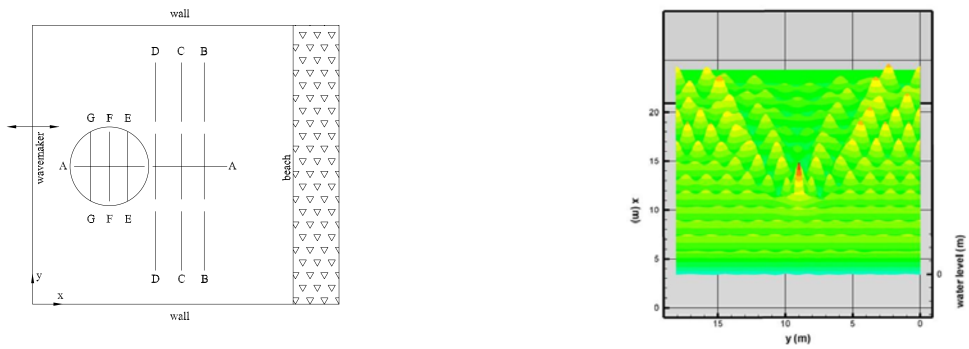

Figure 2: Wave basin (left) and Snapshot at $t = 33$s (right)

On Figure 3, along the longitudinal transect A-A (y =8.98 m), the G-N model predicts very well the wave shoaling and focusing and the decrease of wave height after the shoal. The wave height at the right end of transect A-A is zero, and this shows that the numerical beach works very well. The main difference between G-N model and Boussinesq model is shown in transect F-F. From F-F, we found that the results of G-N model in the central parts are much better than Boussinesq model compared with experimental data.

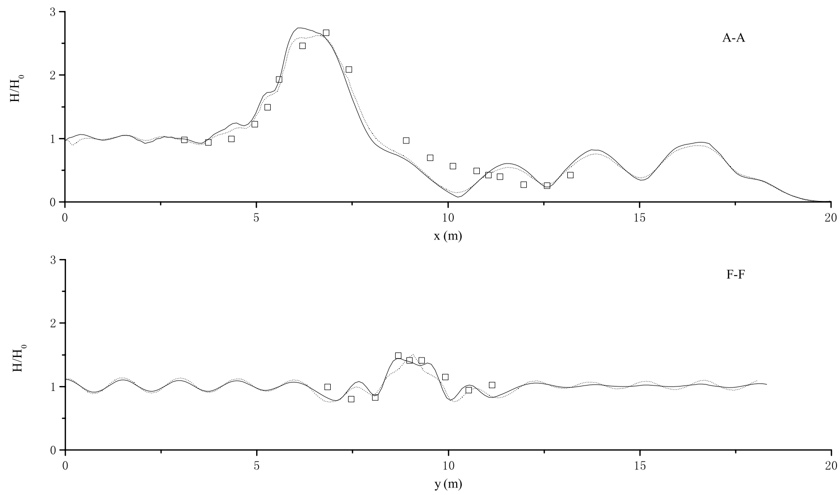

Figure 3: Comparisons of relative wave height. Solid lines: Level II G-N Model; Dot lines: Boussinesq model (Chen et al. 2000); Squares: Experimental data (Chawla and Kirby, 1996)

The second case concerns to simulate oblique waves with $\alpha = 15°$ in deep water. The wave period is T=4s and wave height 2.0m. There are three damping zones with a length of 4λ at the right end, the upside and the downside of the domain. Apart from the damping zone, the waves in the effective zone at 100s are presented on the left side of Figure 4.

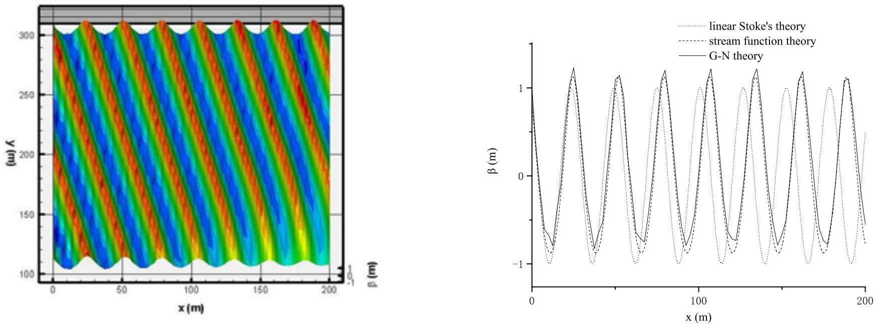

Figure 4: Effective region at 100s (left) and Comparison of wave elevations along $y = 200$m (right)

Finally, the movement of submarine slumps and slides in two orthogonal directions is researched here. Figure 5 presents tsunami waveforms, β/ζ_0, for two-dimensional motion of the slides and slumps in the case of $L \times W = 50 \times 50$ km^2, $h = 2$ km, v_T/v=0.1, $v^2 = v_1^2 + v_2^2$, $v_1/v_2 = 1$ and $t = t^* = L/v_1$.

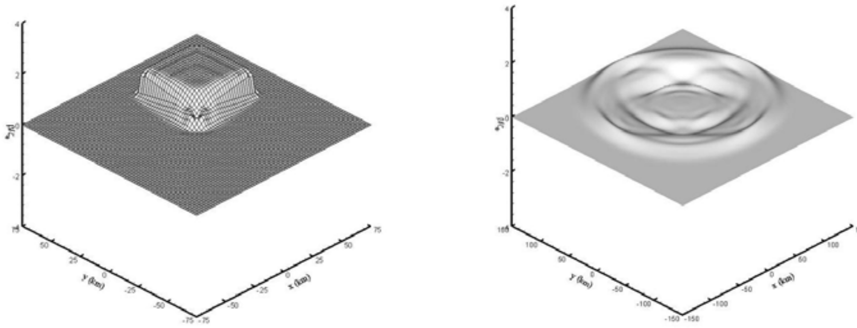

Figure 5: Tsunami wave forms (β/ζ_0)
at $t = t^* = L/v_1$=0.8min (left), and t =11.77min (right)

4. Conclusions

A fully nonlinear, high dispersive three dimensional G-N model is presented. The difference of dispersive relation between Level VII G-N theory and the linear Stokes wave theory reached $O[(kh)^{17}]$, and Level VII G-N theory can predict the waves with $kh \leq 33$, but the highest derivative is only 3-rd. It can predict the wave deformation caused by bottom deformation, including earthquake-induced tsunami. Also, it can simulate fully nonlinear deep waves. The G-N models introduced here provide a very useful numerical tool for investigating nearshore and offshore waves.

Acknowledgments

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