Hybrid finite difference/BEM solutions of the elliptic mild slope equations

Roohollah Naseri Zadeh
Dept. of Civil Engineering
Tehran University
E-mail: naserizdh@ut.ac.ir

Harry B. Bingham
Dept. of Mechanical Engineering
Technical University of Denmark
hbb@mek.dtu.dk

Introduction

This abstract describes a numerical solution of the elliptic mild slope equations. The elliptic mild slope equations are suitable for linear analysis of wave diffraction around fixed, bottom mounted coastal structures. Within the context of linear theory, they capture all refraction, and scattering effects and in modified form can accurately treat sea bed slopes of order one or more. Modifications also exist for modelling partial reflection and wave dissipation due to wave breaking and bottom friction. Since the equations treat the full three-dimensional problem by solving a Helmholtz-type equation on a two-dimensional plane, the model has the potential to analyse large sections of a coastal region with a relatively small computational effort.

The numerical solution of the elliptic mild slope equations described here is based on a combination of the finite difference method (FDM) and the boundary element method (BEM). In the *interior* region, where water depth effects are important and scattering structures exist, we apply a flexible order finite difference method on overlapping boundary-fitted blocks. This model can be used alone, with numerical incident and radiation boundary conditions applied, or it can be coupled to a BEM model in the far-field where the depth is assumed to be constant and scattered waves should be allowed to radiate to infinity.

The model is still in the early stages of development, so results are presented here for several validation test cases which begin to establish its characteristics and performance.

Formulation

We adopt a Cartesian coordinate system with origin at the still water level and the z-axis vertically upward. The horizontal coordinate is $\mathbf{x} = (x, y)$, the still water depth is given by $h(\mathbf{x})$ and $\boldsymbol{\nabla} = (\partial_x, \partial_y)$ is used for the horizontal gradient operator. The gravitational acceleration $g = 9.81 m^2/s$ is assumed to be constant. The linearized potential flow solution is taken to be of the form

$$\Phi(\mathbf{x}, z, t) = \Re\{\varphi(\mathbf{x}, t) f(z)\}, \quad \varphi(\mathbf{x}, t) = \phi(\mathbf{x}) e^{\mathrm{i}\omega t}, \quad f(z) = \frac{\cosh\left[k(z+h)\right]}{\cosh\left(kh\right)} \tag{1}$$

where k and ω are the wave number and frequency respectively which are related by the linear dispersion relation $\omega^2 = gk\tanh(kh)$. The time-harmonic, modified mild slope equation of [1] can be used to solve for ϕ:

$$\boldsymbol{\nabla} \cdot (I_1 \boldsymbol{\nabla}\phi) + (k^2 I_1 + r)\phi = 0 \tag{2}$$

where

$$r = I_2 \nabla^2 h + (\boldsymbol{\nabla} h)^2 \left(\frac{\partial I_2}{\partial h} - I_3 \right) \tag{3}$$

with

$$I_1 = \int_{-h}^{0} f^2 dz, \quad I_2 = \int_{-h}^{0} f \frac{\partial f}{\partial h} dz, \quad I_3 = \int_{-h}^{0} \left(\frac{\partial f}{\partial h}^2 \right) dz. \tag{4}$$

As shown by [4], (2) can be put into Helmholtz form as

$$\nabla^2 \psi + k_c \psi = 0 \tag{5}$$

where

$$\psi = \sqrt{I_1}\phi, \qquad k_c^2 = k^2 + \frac{r}{I_1} - \frac{\nabla^2 \sqrt{I_1}}{\sqrt{I_1}}, \tag{6}$$

which is convenient for numerical solution. Dissipation terms to model wave breaking and bottom friction effects can also be Incorporated into the equation as in *e.g.* [5].

Numerical Solution

Our numerical solution is similar to that presented by [5] and [2] in that we solve (5) by means of finite differences. In those references however, the grid is strictly Cartesian with a uniform grid spacing in each direction and derivatives are approximated to second-order accuracy. Here we develop the solution on boundary-fitted, overlapping blocks and discretize the derivatives to arbitrary order. We also introduce a matching to a BEM method at the far-field boundary of the computational domain as an alternative means of implementing the incident wave and radiation boundary conditions.

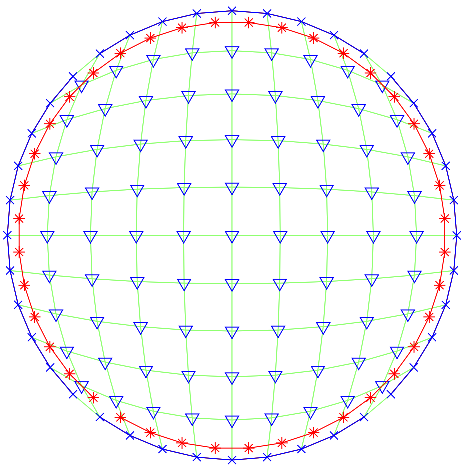

Figure 1: A circular domain grid example. Triangles are interior grid points where (5) is enforced; stars indicate the BEM boundary panel vertices; and crosses are grid points used to enforce continuity of the two solutions.

Figure 1 shows a sample discretization of the problem with a circular interior domain boundary. The stars on this figure indicate the BEM panel vertices where the boundary conditions are imposed. The crosses indicate finite difference grid points which are used to impose continuity of the FDM and BEM solutions on the BEM boundary by interpolation of the FDM solution onto the BEM panel collocation points. Continuity of normal flux through the BEM boundary is imposed via the BEM.

In the *exterior* computational domain we assumed a constant fluid depth so that the scattered wave potential ϕ^S satisfies the Helmholtz equation

$$\nabla^2\phi^S + k_0^2\phi^S = 0 \tag{7}$$

where k_0 is the wavenumber in the outer domain. The fundamental solution of (7) is $\phi^*(\xi, \mathbf{x}) = (\mathrm{i}/4)H_0^{(1)}(k_0 R)$ where $R = |\xi - \mathbf{x}|$ is the distance between the source and field points and $H_0^{(1)}$ is the Hankel function of the first kind of zero order. Multiplying both sides of (7) by ϕ^* and using Green's second identity, together with the Sommerfeld radiation condition at infinity, we can rewrite (7) as

$$-c_\xi(\xi)\phi^S(\xi) + \int_{\Gamma_0}\left(\frac{\partial\phi^S}{\partial n}\phi^* - \phi^S\frac{\partial\phi^*}{\partial n}\right)d\Gamma = 0 \tag{8}$$

where $c_\xi(\xi)$ is the included angle at a collocation point ξ on the boundary Γ_0 and $\partial/\partial n$ is the derivative in the direction normal to the boundary.

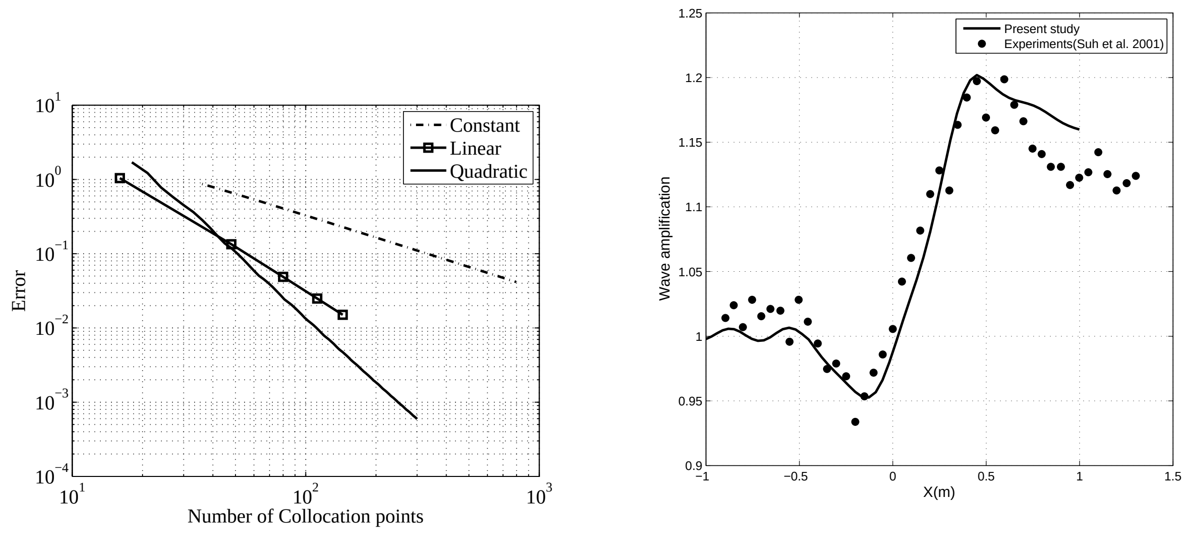

Figure 2: Left: Maximum run-up errors for diffraction around a circular cylinder plotted against total number of BEM unknowns. Right: Diffraction around a circular shoal compared to [6].

To solve (8), we apply iso-parametric approximations using constant, linear or quadratic elements. To validate the BEM solution, we first consider diffraction around a bottom mounted circular cylinder in constant depth water, *i.e.* Γ_0 is treated as a solid boundary. The numerical solution for run-up around the cylinder can be compared with the exact solution of [3]. On the left of Figure 2, we plot the maximum relative error as a function of the total number of unknowns in From the plot it is clear that the methods converge asymptotically at the expected rates and are first-, second- and third-order accurate respectively. This plot also shows that linear and quadratic elements are always more efficient than constant elements and that to ensure errors of less than about 10^{-1}, quadratic elements are most efficient.

To match the BEM with the FDM solution in the interior domain, we insert $\phi^S = \phi - \phi^I$ into (8) where ϕ^I is the known incident wave in the exterior region. The known terms are moved to the right hand side, and $\partial\phi/\partial n$ at each collocation point of Γ_0 is expressed in terms of FDM derivatives using nearby grid points. These BEM equations are then added to the FDM equations discussed above to give a linear system of size $N_{fdm} + N_{bem}$. The system is then solved using direct sparse matrix algorithms.

On the right hand plot of Figure 2, we show some preliminary calculations of the wave elevation amplification factor for waves travelling over a circular shoal. The calculations are compared to the experimental data based of [6] for $k_0 h_0 = 3$. Further results will be presented at the workshop.

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