THE EXISTENCE AND NON-EXISTENCE OF WAVELESS SHIPS

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Question: Consider two-dimensional flow past a surface-piercing ship. At low (draft-based) Froude numbers, is it possible to design the hull in such a way as to minimize or entirely eliminate the waves produced by the ship?

1 In Search for a Waveless Ship (1970s-1990s)

Our story begins in 1968 [1], when naval architect T.F. Ogilvie noticed several peculiarities with asymptotic predictions of ideal two-dimensional slow flows over an obstruction; this describes, for example, flows over a bumpy ocean floor or over a step in a channel.

First, Ogilvie remarked that the approximations predicted a *waveless* free surface, but despite the speed of the stream being small, one would still expect waves to form downstream of the obstruction. *Why had the asymptotics failed in capturing such waves?*

Second, he noted that previous researchers had derived asymptotic expansions in which the asymptotic expansions seem to 're-order' as the speed of the stream tends to zero. So for example, at moderate speeds, a single term in the asymptotic expansion might appear to be an adequate approximation. But for lower speeds, one would need to include a second term to achieve the desired accuracy. For still lower speeds, the third-order terms would become important. And so on.

Today, it is now known that at low Froude numbers, the waves are in fact exponentially small and thus *beyond all orders* of regular asymptotics; their formation is a consequence of the divergence of the asymptotic series and the associated Stokes Phenomenon [2].

This underlying subtlety has been painfully problematic in regards to previous asymptotic and numerical treatments of the nonlinear ship-wave problem. In [3], Dagan and Tulin showed that the analysis near a three-dimensional ship can be reduced to studying the two-dimensional ideal flow problem where the ship is modeled as a semi-infinite body with constant draft. This fully nonlinear free-surface problem was first computed by Vanden-Broeck and Tuck [4], and on the basis of numerical evidence, they conjectured that ship hulls with a single front face would always generate waves (see Figure 1).

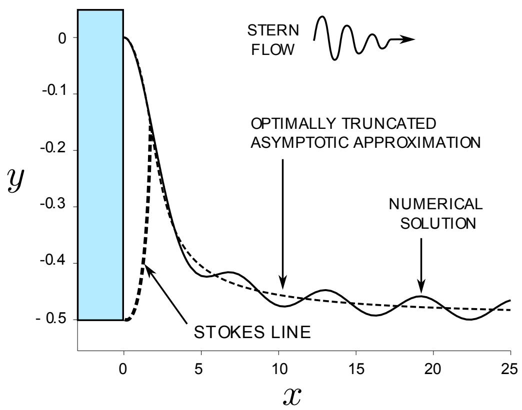

FIGURE 1: The *Low-Speed Paradox* addresses the fact that that asymptotic expansions in powers of the Froude number fail to capture waves present on the free surface in potential flow problems. In this figure, the asymptotic (waveless) expansion of stern flow is compared to the numerical solution of the problem. The key idea is that a Stokes line emerges from the corner-singularity, across which an exponentially small wave turns on.

Moreover, the earlier experimental work of Baba [5] had indicated that a bulbous bow can eliminate, or at least reduce the splash at the bow of a ship[1]. This prompted the discovery of seemingly waveless ships with bulbous profiles, first by Tuck and Vanden-Broeck [6] and later confirmed by Madurasinghe [7]—but again, only *numerically* so. Unfortunately, these results were later refuted by the more comprehensive numerical study of Farrow and Tuck [8]; there, they wrote that

> The free surface would at first sight appear to be waveless, but on closer examination of the numerical data, there are very small waves present.

Clearly, these are questions which cannot be easily answered using simple numerics. Indeed in [9], Tulin mentions two open questions:

> The fundamental questions of whether such rising potential free-surface flows before bluff bodies exist [...] still remain open,

and

> Is it demonstrable [...] that continuous solutions will not exist in the limit of vanishing speed? Does this have anything to do with the inability of Tuck and his colleagues [...] to find a continuous solution in the two-dimensional

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[1] In potential flow, a waveless solution past the stern (rear) of a ship is equivalent to a splashless solution at the bow (front) of a ship.

bow wave case? Do nonbreaking flows exist at all for surface-piercing ship forms of arbitrary form and thickness, at any speed?

However, with the recent development of techniques in exponential asymptotics (see for example [10] and [11]), many of these issues can be resolved.

2 Mathematical Formulation

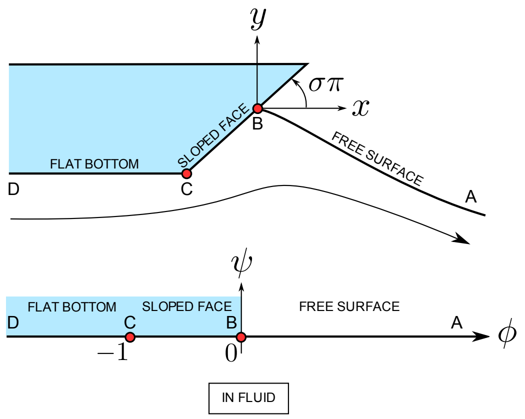

FIGURE 2: The physical (x, y) and potential (ϕ, ψ) planes

Consider steady, two-dimensional, incompressible, inviscid, and irrotational flow past a semi-infinite body consisting of a flat bottom and a face with slope $\sigma\pi$ to the horizontal. The non-dimensionalized problem can be reposed as a problem in the potential plane (See Figure 2),

$$\log q = \log q_0 + \frac{1}{\pi}\int_0^\infty \frac{\theta(\varphi)}{\varphi - \phi}\, d\phi \tag{1}$$

$$\epsilon q^2 \frac{dq}{d\phi} = -\sin\theta, \tag{2}$$

where ϵ is related to the square of the Froude draft number, $\nabla\phi$ is the non-dimensionalized fluid velocity, $q = q(\phi)$ is the speed of the flow, $\theta = \theta(\phi)$ is the angle the streamlines make with the x-axis, and

$$q_0 = \left(\frac{\phi}{\phi+1}\right)^\sigma \tag{3}$$

is the leading-order ($\epsilon = 0$) solution. Here, $\phi = 0$ is chosen to be the stagnation point and $\phi = -1$ to be the corner. Now solving the stern problem is equivalent to solving the above equations for q and θ for $\phi \geq 0$ (the free-surface).

The key is that we will be interested in studying the *analytic continuation* of the free-surface and thus allowing $\phi + i0 \mapsto w$, $q(\phi, 0) \mapsto q(w)$, and $\theta(\phi, 0) \mapsto \theta(w)$ to be elements of a complex variable. Analytically continuing Equations (1) and (2) gives

$$\log q \mp i\theta = \log q_0 + \frac{1}{\pi}\int_0^\infty \frac{\theta(w')}{w' - w}\, dw' \tag{4}$$

$$\epsilon q^2 \frac{dq}{dw} = -\sin\theta, \tag{5}$$

the $\mp$ signs corresponding to analytic continuation into the upper and lower-half planes, respectively.

3 The Simplified Nonlinear Problem

Although the full problem (4)-(5) is tractible within our methodology, for reasons of brevity we will present a simpler problem that nevertheless illustrates the key ideas.

It is the case that when the exponentially small terms are sought from Equations (4), the integral term serves to only change the amplitude coefficient of the downstream waves by a non-zero, $O(1)$ amount. That is, the salient features of the problem are still retained when we use $\log q \mp i\theta = \log q_0$ instead of Equation (4). Without loss of generality, we will choose to analytically continue into the upper-half plane and substituting this simplification into Equation (2) gives

$$\epsilon q_0 q^3 \frac{dq}{dw} - \frac{i}{2}\left(q^2 - q_0^2\right) = 0. \tag{6}$$

Now, why is it that the integral matters so little? As we shall see, the determination of the waves essentially depends on an analysis near the singularities in the analytic continuation of the free surface. However, the boundary integral is evaluated along the free surface ($\phi \geq 0$), *away* from the singularities. Thus for the *full* problem (4)-(5), the integral plays no actual role for much of the analysis.

It can be shown that the question of *existence* of waveless ships is equivalent to solving the simplified problem in Equation (6), even though the *quantitative* results between the two problems are slightly different.

4 Applying Exponential Asymptotics

We present the key ideas. First begin as usual by calculating the regular asymptotic expansion of Equation (6) in the limit $\epsilon \to 0$, letting

$$q = \sum_{n=0}^\infty \epsilon^n q_n. \tag{7}$$

The leading order solution $q_0(w)$ is the rigid-body flow of Equation (3), while the $O(\epsilon)$ expression gives

$$q_1 = -iq_0^3 \frac{dq_0}{dw} \tag{8}$$

and at $O(\epsilon^n)$ for $n \geq 2$,

$$3q_0^4 \frac{dq_{n-1}}{dw} + 3q_0^3 q_{n-1}\frac{dq_0}{dw} + 3q_0^3 \frac{dq_{n-2}}{dw} + \ldots - i\Big(q_0 q_n + q_1 q_{n-1} + \ldots\Big) = 0. \tag{9}$$

The crucial observation is that there exists a *singularity* in the analytic continuation of q_0 at $w = -1$, the corner of the stern. This use of *ill-defined* approximations in order to represent perfectly well-defined phenomena is one of the caveats of asymptotics, but one would feverishly hope that a singularity far from the region of interest (the free-surface) has little effect on the approximation!

Unfortunately, this is not the case. We can see from Equation (9) that at each order, q_n is partially determined by differentiating q_{n-1} once; thus each additional order adds to the power of the singularity in q_0. As $n \to \infty$, the analytically continued asymptotic expansion (7) will exhibit factorial over power divergence in the form

$$q_n \sim \frac{Q(w)\Gamma(n+\gamma)}{[\chi(w)]^{n+\gamma}}, \qquad \text{as } n \to \infty \tag{10}$$

where γ is a constant, and $Q(w)$ and $\chi(w)$ are functions to be determined. Thus, the unsettling growth of the factorial is expounded by the fact that the power of the singularity, $\chi(-1)$ grows at each subsequent order. The late terms are therefore entirely *dominated* by the singularity at the corner.

Now, an asymptotic analysis in the limit $n \to \infty$ using the above ansatz reveals

$$\chi(w) = \int_{-1}^{w} \frac{-i}{[q_0(s)]^3}\, ds \tag{11}$$

and

$$Q(w) = \frac{\Omega e^{i\Phi}}{2(1+3\sigma)^{\gamma}[q_0(w)]^5}, \tag{12}$$

where Ω and Φ are constants. The determination of γ, Ω, Φ, and in fact, the Stokes line smoothing in the next section will require an analysis near the singularity.

First, since by Equation (3), $q_0 \sim c(w+1)^{-\sigma}$ where c is constant, we must require—by comparison of powers in the components of Equation (10)—that $\gamma = 6\sigma/(1+3\sigma)$. Second, in order to determine the constant Ω, we need to re-scale near the singularity, express the leading-order inner solution as a power series (in inner coordinates) and match with the outer solution. In the end, however, Ω is determined by the numerical solution to a nonlinear reccurence relation. The values of Ω for various values of σ are shown in Figure 3.

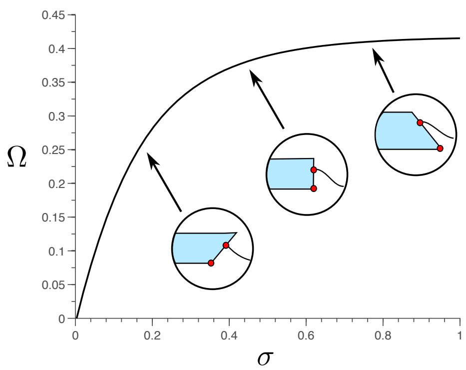

FIGURE 3: Notice that $\Lambda \neq 0$ for all one-cornered ships. This guarantees the non-existence of a waveless ship.

5 Smoothing the Stokes Line

The underlying divergence of the asymptotic expansions will cause the *Stokes Phenomenon* to occur: as the complexified asymptotic solution crosses a critical line (the *Stokes Line*), a small exponential switches on. When Stokes first studied this phenomenon in the context of the Airy Equation, he wondrously described the exponential as seeming to emerge from behind a mist.

In order to identify the exponentially small waves, we *optimally truncate* the asymptotic series at $n = N$ so that

$$q = \sum_{n=0}^{N} \epsilon^n q_n + R_N. \tag{13}$$

At the optimal truncation point, the remainder R_N is *exponentially small* (rather than only algebraically small) and can be written as $R_N(z) = S(z)Qe^{-\chi/\epsilon}$, where we expect $S(z)$ to smoothly vary from zero to a constant across the Stokes Line. As shown by Dingle [12], Stokes Lines can be expected wherever $\chi(w)$ is *real and positive*.

The procedure then is to re-scale near the critical line and examine the jump in the exponentially small remainder as the Stokes Line is crossed. The jump in the remainder is shown to be

$$\Big[R_N\Big] \sim \frac{2\pi Q(z)}{\epsilon^{\gamma}} \exp\Big[-\frac{\chi}{\epsilon}\Big], \tag{14}$$

and thus, after some work, the amplitude of the exponentially small waves is revealed to be

$$q_{\text{exp}} \sim \frac{\Omega\pi \exp\left[\frac{-3\pi\sigma}{\epsilon}\right]}{\epsilon^{\gamma}(1+3\sigma)^{\gamma} q_0^5} \exp\left[i\frac{\Im(\chi)}{\epsilon} + i\Phi\right] \tag{15}$$

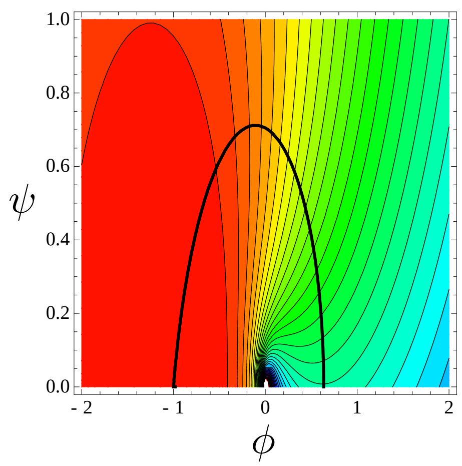

FIGURE 4: Contour plot for $|\chi(w)|$. The thick black line is the Stokes Line (compare with Fig. 1). Near the singularity ($w=-1$), χ is small and thus the resultant exponential is $O(1)$. As we travel along the Stokes line towards the free-surface, $\Re[\chi]$ grows until it is indeed exponentially small on the free-surface.

6 Numerical Comparisons

Although at first sight, Equation (6) appears to be a relatively simple first-order differential equation, care must be taken in order to ensure that the solution is properly resolved near the stagnation point $w = 0$. To do this, we (1) apply the coordinate transformation $s = w^\sigma$, and (2) apply the boundary condition at $w = 0$ as $q(L) = 0$, and verify that the solutions are numerically convergent as $L \to 0$.

In Figure 5, we plot the numerical amplitudes (of either the real or complex parts of q) over a range of Froude numbers and for a range of angled sterns. The agreement between analytical prediction and numerical computation is remarkably good, even for a relatively large Froude number.

Numerical computation of the full nonlinear problem in Equations (4)-(5) is more difficult and possesses many subtleties. However, the agreement between analytical and numerical results are still very good.

7 Discussion

So in the end, *do waveless ships exist?*

For the one-cornered ship, the answer is quite clearly *'No!'* The Stokes line smoothing *necessitates* the existence of a non-zero wave (15) on the free-surface (since the pre-factor Ω is non-zero for all values of $0 < \sigma < 1$). For stern flows, these waves must propagate downstream, while for bow flows, these waves grow to be of infinite amplitude near the hull[2]. Thus for the one-cornered hull, neither waveless sterns nor splashless bows are possible.

And for more general ships? *Perhaps.* Our theory has been successively implemented for more general piecewise-linear hull forms; here, analytical criteria can be provided for the construction of waveless ships. With several corners, it may be possible (though difficult) to produce total phase cancellation. Work on the bulbous profiles considered by Tuck and others is ongoing.

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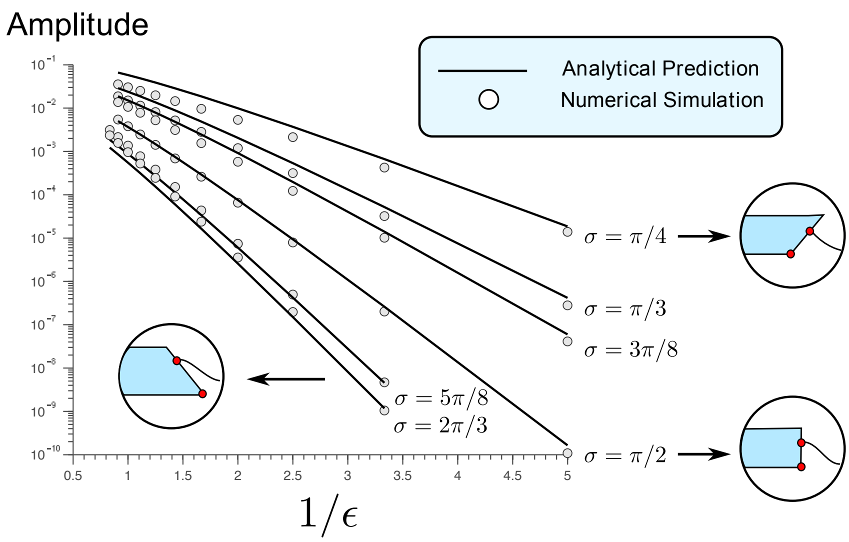

FIGURE 5: Asymptotic (line) and numerical (circle) amplitudes. The straight-line fit on the semi-log scale indicates the exponentially small nature of the stern waves.

[2]In particular, this implies that for bow flows, the assumption that the flow attaches to the hull at a stagnation point is *false*. In [4], Vanden-Broeck and Tuck conjecture that the correct assumption in the low-Froude limit must include an overturning splash.