RADIATION LOADS ACTING ON A HORIZONTAL CYLINDER OSCILLATING IN STRATIFIED FLUID WITH AN ICE-COVER

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1. Introduction

The effect of density stratification on the hydrodynamic loads acting on an oscillating submerged body was considered for some particular cases of stratified fluid (see, for example, the reference book by Korotkin, 2009). The fluid was bounded above by either a free surface or a rigid plate. The investigation of ice-wave interaction problem is of interest for the polar oceans covered with ice. To the author's knowledge, the effect of an ice-cover was investigated only for wave radiation by a submerged sphere in homogeneous water (Das & Mandal, 2008).

In this paper, the linear 2-D water-wave problem describing small oscillations of a horizontal cylinder is considered. The cylinder is submerged in a uniformly stratified fluid with flat bottom. The fluid is bounded above by a layer of ice-cover modelled as a thin elastic sheet.

2. Statement of the problem

Let a Cartesian coordinate system be taken with the x-axis directed along the undisturbed position of the ice-cover perpendicular to the cylinder axis, and the y-axis pointing vertically upwards. The fluid is assumed to be both inviscid and incompressible. The fluid density $\rho_0(y)$ increases linearly with depth: $\rho_0(y) = \rho_s(1 - \alpha y)$, where $\alpha > 0$ and $\rho_s = \rho_0(0)$. The fluid depth is equal to H. The wave motions are initiated in the fluid, which is initially at rest, by the small oscillations of a cylinder at a frequency ω with amplitudes η_j $(j = 1, 2, 3)$ for the sway, heave and roll problems, respectively.

Under the usual assumptions of linear theory, a disturbed pressure in the fluid can be written as

$$P(x, y, t) = \rho_s \mathrm{Re}\Big[\exp(i\omega t) \sum_{j=1}^{3} \eta_j p_j(x, y)\Big],$$

where $p_j(x, y)$ are complex valued functions and t is time. In the Boussinesq approximation, the function $p_j(x, y)$ obeys the equation (see, for example, Miropol'sky, 2001)

$$\nabla^2 p_j = \frac{N^2}{\omega^2}\frac{\partial^2 p_j}{\partial x^2} \quad (-\infty < x < \infty,\ -H < y < 0), \quad N = \sqrt{-\frac{g}{\rho_s}\frac{d\rho_0}{dy}} = \sqrt{\alpha g} \tag{1}$$

except in the region occupied by the cylinder. Here $N = const$ is the buoyancy frequency and g is the acceleration due to gravity.

The linearized ice-cover condition is

$$\Big(B\frac{\partial^4}{\partial x^4} + Q\frac{\partial^2}{\partial x^2} + g - M\omega^2\Big)\frac{\partial p_j}{\partial y} + (N^2 - \omega^2)p_j = 0 \quad \text{on } y = 0, \tag{2}$$

where

$$B = \frac{Eh_1^3}{12(1-\nu^2)\rho_s}, \quad M = \frac{\rho_1 h_1}{\rho_s},$$

E is the Young's modulus for the ice, ν is its Poisson's ratio, Q is the compressive force, ρ_1 is the density of the ice and h_1 is the small thickness of the ice-cover. When the flexural rigidity B and the compressive force Q are taken to be zero, so that the ice sheet behaves as a floating set of disconnected mass points (the broken ice). When in addition also surface density of ice-cover M is taken to be zero, then upper boundary of fluid becomes a free-surface.

The boundary condition on the closed smooth contour of the submerged body S has the form:

$$n_x\frac{\partial p_j}{\partial x} - \frac{n_y}{\beta^2}\frac{\partial p_j}{\partial y} = \omega^2 n_j \quad (x, y \in S), \quad \beta^2 = \frac{N^2}{\omega^2} - 1. \tag{3}$$

Here, $\mathbf{n} = (n_x, n_y)$ is the inward normal to the contour S. The notations

$$n_1 = n_x, \quad n_2 = n_y, \quad n_3 = (y - y_0)n_1 - (x - x_0)n_2$$

are used where $x_0,\ y_0$ are the coordinates of the center of the roll oscillations.

The boundary condition at the bottom is

$$\frac{\partial p_j}{\partial y} = 0 \quad \text{on} \quad y = -H. \tag{4}$$

In the far field a radiation condition should be imposed that requires the radiated waves to be outgoing.

The radiation load acting on the oscillating body is determined by the force $\mathbf{F} = (F_1, F_2)$ and the moment F_3 which, without account for the hydrostatic term, have the form

$$F_k = \sum_{j=1}^{3} \eta_j \tau_{kj}, \quad \tau_{kj} = \rho_s \int_S p_j n_k ds = \omega^2 \mu_{kj} - i\omega\lambda_{kj} \quad (k = 1, 2, 3), \tag{5}$$

where μ_{kj} and λ_{kj} are the added mass and damping coefficients, respectively.

The behavior of the solution of the radiation problem (1)-(4) depends significantly on the body oscillation frequency. When $\omega < N$ ($\beta^2 > 0$) equation (1) is hyperbolic and the oscillations of the body generate both surface and internal waves in the fluid. When $\omega > N$ ($\beta^2 < 0$) equation (1) becomes elliptic and only the surface wave may be generated. In what follows, these cases will be considered separately.

3. Case $\omega < \mathbf{N}$

In order to solve problem (1)-(4) we introduce an unknown mass-source distribution $\sigma_j(x, y)$ over the contour S. We can now represent the pressure at any point of the fluid in the form:

$$p_j(x, y) = \int_S \sigma_j(\xi, \eta) G(x, y; \xi, \eta) ds. \tag{6}$$

Here, $G(x, y; \xi, \eta)$ is the Green function of the problem, which determines the disturbed pressure in the fluid initiated by an oscillating mass source with unit strength, where (x, y) is the field point and (ξ, η) is the source point. The Green function must satisfy the following equation

$$\frac{\partial^2 G}{\partial y^2} - \beta^2 \frac{\partial^2 G}{\partial x^2} = 2\pi\delta(x - \xi)\delta(y - \eta)$$

with the boundary conditions analogous to (2), (4) and the radiation condition in the far field, and δ is the Dirac delta-function. Using traditional Fourier techniques, one may find the solution as

$$G = -\frac{i\pi}{\beta^2} \sum_{n=0}^{\infty} \frac{\cos k_n\beta(y + H)}{k_n D_n} \cos k_n\beta(\eta + H) \exp(-ik_n|x - \xi|),$$

where k_n $(n = 0, 1, 2, ...)$ are the real positive roots of the equation

$$\tan(k\beta H) = C/[k\beta\Lambda(k)], \quad C = N^2 - \omega^2, \ \Lambda(k) = Bk^4 - Qk^2 + g - M\omega^2, \tag{7}$$

$$D_n = [\sin(2k_n\beta H)/(2k_n\beta) + H]/2.$$

The equation (7) is the dispersion relation which gives different wavenumbers k_n $(k_n < k_{n+1})$ for a given frequency ω. The smallest wavenumber k_0 is referred to as the surface-wave mode propagating along the ice-cover. All other wavenumbers k_n $(n \geq 1)$ are referred to as the internal-wave modes which exist only at $\omega < N$ with $k_n \to \infty$ as $\omega \to N$.

Using the body boundary condition (3) we obtain the integral equation for determining the function $\sigma_j(x, y)$

$$\pi\sigma_j(x, y) - \int_S \sigma_j(\xi, \eta) \left[n_1 \frac{\partial G}{\partial x} - \frac{n_2}{\beta^2} \frac{\partial G}{\partial y} \right] ds = \omega^2 n_j.$$

After calculating the distribution of the singularities $\sigma_j(x, y)$, we can determine the pressure (6) and the hydrodynamic load (5).

4. Case $\omega > \mathbf{N}$

In this case we can also use the distributed-singularity method but for the uniformly stratified fluid it is more convenient to solve the integral equation for the pressure. Now the equation (1) has the form

$$\frac{\partial^2 p_j}{\partial x^2} + \frac{1}{\gamma^2}\frac{\partial^2 p_j}{\partial y^2} = 0, \quad \gamma^2 = -\beta^2 = 1 - \frac{N^2}{\omega^2}. \tag{8}$$

If we introduce the transform $\bar{y} = \gamma y$ of the vertical coordinate, then in the coordinate system x, $\bar{y}$ the equation (8) reduces to the Laplace equation and the body boundary condition (3) reduces to the value of the normal derivative on the deformed contour, correct to a multiplier depending on the body geometry. This made it possible to use the affine similitude for determining the hydrodynamic load acting on an arbitrary contour oscillating in an unbounded uniformly stratified fluid (Ermanyuk, 2002).

Using the Green identity, the boundary conditions (2), (4) and the radiation condition in the far field, we obtain the integral equation which for points located on the transformed body contour $\bar{S}$ has the form:

$$p_j(x, \bar{y}) = \frac{1}{\pi}\int_{\bar{S}} \left[p_j(\xi, \bar{\eta}) \frac{\partial G}{\partial \bar{n}} - G(x, \bar{y}; \xi, \bar{\eta}) \frac{\partial p_j}{\partial \bar{n}} \right] d\bar{s}. \tag{9}$$

Here, the bar denotes the values considered in transformed coordinates.

The Green function $G(x, \bar{y}; \xi, \bar{\eta})$ has the form:

$$G = \ln \frac{\bar{r}}{\bar{r}_1} + pv \int_0^\infty \Big[A(k, \bar{\eta}) \exp(-k\bar{y}) + B(k, \bar{\eta}) \exp(k\bar{y}) \Big] \frac{\cos k(x - \xi)}{T(k)} dk -$$

$$i\pi \Big[A(k_0, \bar{\eta}) \exp(-k_0\bar{y}) + B(k_0, \bar{\eta}) \exp(k_0\bar{y}) \Big] \frac{\cos k_0(x - \xi)}{T'(k_0)} \tag{10}$$

where pv indicates the principal-value integration,

$$\bar{r} = \sqrt{(x - \xi)^2 + (\bar{y} - \bar{\eta})^2}, \quad \bar{r}_1 = \sqrt{(x - \xi)^2 + (\bar{y} + \bar{\eta})^2},$$

$$A(k, \bar{\eta}) = \left[\frac{C}{k} - \gamma\Lambda(k) \right] e^{k(\bar{\eta} - 2\bar{H})} - \left[\frac{C}{k} + \gamma\Lambda(k) \right] e^{-k(\bar{\eta} + 2\bar{H})},$$

$$B(k, \bar{\eta}) = -\left\{ 2\gamma\Lambda(k) e^{k\bar{\eta}} + \left[\frac{C}{k} - \gamma\Lambda(k) \right] [e^{k(\bar{\eta} - 2\bar{H})} - e^{-k(\bar{\eta} + 2\bar{H})}] \right\},$$

$$T(k) = C + \gamma k\Lambda(k) + [C - \gamma k\Lambda(k)] e^{-2k\bar{H}}, \quad T'(k_0) \equiv dT/dk|_{k=k_0}.$$

The value k_0 is the unique positive root of the equation

$$T(k_0) = 0. \tag{11}$$

This root corresponds to the existence of the flexural-gravity wave at $B \neq 0$ or the surface wave at $B = Q = 0$. However, the equation (11) may not have the positive root in specific cases, for example, at $\omega > \sqrt{g/M}$ for the broken ice (Kheysin, 1967). Then the last term in (10) should be omitted.

As a result of solving the integral equation (9), we can determine the pressure distribution along the body contour and then the corresponding hydrodynamic load.

5. Numerical results

Numerical calculations are performed for a circular cylinder of radius a whose cross-section S is given by the equation $x^2 + (y + h)^2 = a^2$, where h is the distance from the center of the cylinder to the mean upper boundary of the fluid. The following numerical input data are used:

$E = 5{\times}10^9 Pa$, $\rho_s = 1025\, kg\, m^{-3}$, $\rho_1 = 922.5\, kg\, m^{-3}$, $h_1 = 2\, m$, $Q = 10^4 N\, m^{-2}, \nu = 0.3, N = 0.05\, s^{-1}$,

$$a = 10\ m, h = 20\ m, H = 500\ m.$$

Fig. 1 shows the dispersive curves for the surface wave mode k_0 and the first five internal wave modes $(k_1, ..., k_5)$. The solid lines correspond to the ice sheet, the dashed line is for the broken ice and the dash-dotted line is for the free surface. Distinction between these three cases occurs only for relatively high frequency $\omega > 10N$.

The added-mass and damping coefficients are plotted in Figs. 2, 3. The open and dark symbols correspond to the hydrodynamic coefficients for sway and heave, respectively: the circles are for the ice sheet, the triangles are for the broken ice and the squares are for the free surface. The value $\omega = N$ is shown by the arrow in Figs. 2, 3.

More detailed results will be presented at the Workshop.

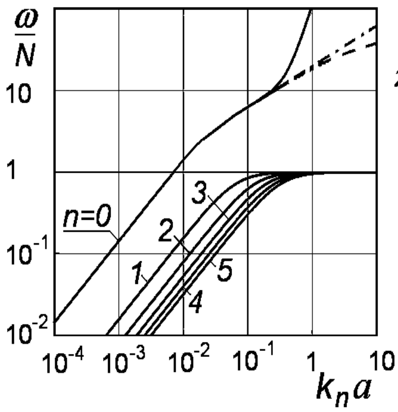

Figure 1

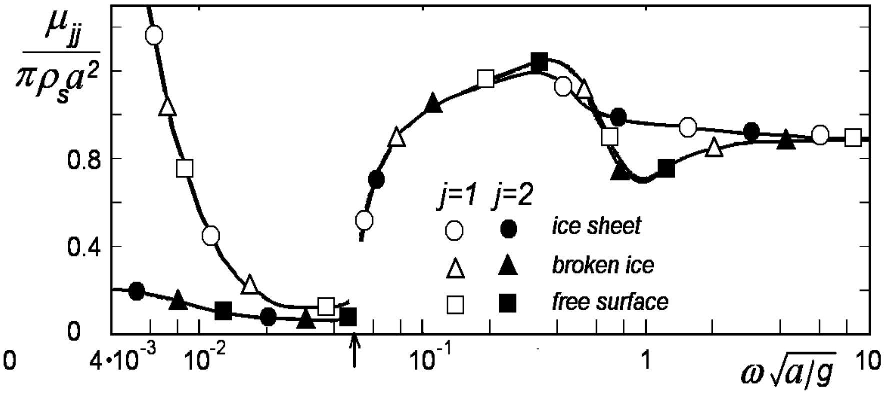

Figure 2

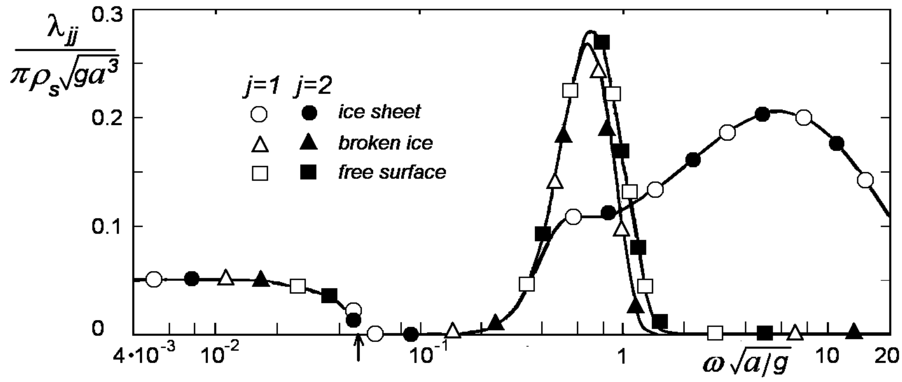

Figure 3

Acknowledgment

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References

Das, D., Mandal, B.N., 2008, Water wave radiation by a sphere submerged in water with an ice-cover, *Arch. Appl. Mech.*, 78(8), 649-661.

Ermanyuk, E.V., 2002, The rule of affine similitude for the force coefficients of a body oscillating in a uniformly stratified fluid, *Experiments in Fluids*, 32(2), 242-251.

Kheysin, D.Ye., 1967, *Dynamics of the Ice Cover*, Gidrometeoizdat, Leningrad, 215 p. (in Russian) Technical Translation FSTC-HT-23-485-69, U.S. Army foreign Science and Technology Center.

Korotkin, A.I., 2009, *Added Masses of Ship Structures*, Ser.: Fluid Mechanics and its Applications, vol. 88, Springer, 392 p.

Miropol'sky, Yu.Z., 2001, *Dynamics of Internal Gravity Waves in the Ocean*, Kluwer Academic Publishers, 428 p.