Some numerical aspects of nonlinear free surface motions by a Method of Fundamental Solutions.

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1) Introduction

Since the pioneering works by Kupradze and Aleksidze (1963), the Method of Fundamental Solutions (MFS also known as Desingularized Methods) has received much attention, both in terms of numerical analysis or in terms of applications. This method applies as soon as there exist exact solutions of the Partial Differential Equation (PDE) driving the Boundary Value Problem (BVP).

Here we focus on the twodimensional interior Dirichlet elliptic BVP as it is posed for the potential flow in a tank with a free surface. The two boundary conditions (kinematic and dynamic) form a time differential system to be solved so that the potential on the free surface is calculated while it deforms.

The principle of MFS is to place singularities (represented by the Green function of the PDE) at a certain distance from the actual boundary. The potential which represents the flow is hence expressed as a finite sum of these singularities with unknown intensities. To compute these intensities, the obtained algebraic equations are classicaly solved via Gauss algorithm or possibly Least square algorithm and a Single Value Decomposition depending on the type of boundary conditions.

Recently a special issue of Engineering Analysis with Boundary Elements (Volume 33, Issue 12, 2009) collects some papers in which numerical difficulties are still underlined. Among them, there is the arbitrary choice of the distance at which the singularities are located. It appears that this distance depends strongly on the application. But it has a crucial influence on the accuracy and (ill-)conditioning of linear system to be inverted.

This abstract highlights some numerical aspects of the MFS applied to the standard nonlinear free surface motions. In particular we focus on the role of the additive constant to the velocity potential in the accuracy and stability of the computed solutions.

Both weakly and highly nonlinear motions are examined ranging from a mild standing wave, up to the flip through phenomenon.

2) Numerical analysis

The governing equations are posed in a coordinate system (O,x,y) whose origin O is the left corner of a rectangular tank. On one side we have the time differential system for the velocity potential ϕ and the lagrangian position $\vec{M}(x,y,t)$ of a marker attached to the free surface. The lagrangian formulation of the kinematic and dynamic boundary conditions reads

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{2}\vec{\nabla}^2\phi - g(y-h), \qquad \frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \vec{\nabla}\phi. \tag{1}$$

where g is the acceleration of gravity and h is the mean fluid level. On the other side, we have the Dirichlet BVP for the velocity potential

$$\Delta\phi = 0 \quad \text{in the fluid}, \quad \phi = e(\vec{M},t) \quad \text{on the fluid contour}. \tag{2}$$

The solution of (2) is broken down as follows

$$\phi(x,y,t) = \sum_{j=1}^{N} q_j(t) G(x,y,X_j(t),Y_j(t)) \tag{3}$$

where (X_j,Y_j) are the source location and q_j is the strength of source j. At a given instant t of the resolution, we know $\phi(x,y,t)$ at discrete points (x,y) along the free surface, say (x_i,y_i) the i^{th} point. By using equation (3) written at N points (x_i,y_i), we get a linear system whose i^{th} line reads

$$\phi_i = \sum_{j=1}^{N} G_{ij} q_j + a(t), \quad \text{with} \quad \left\{ \begin{array}{lcl} \phi_i & = & \phi(x_i,y_i,t) \\ G_{ij} & = & G(x_i,y_i,X_j,Y_j) \end{array} \right. \tag{4}$$

where $a(t)$ is an arbitrary additive constant which does not depend on spatial variables. Its role and influence are discussed in the sequel but its presence is perfectly justified since the right hand side of the Cauchy problem (1) implies only spatial derivatives of ϕ. On the other hand, its time derivative may have some connection with Bernoulli constant. From the litterature the role of $a(t)$ is still an open problem. From the mathematical point of view, Christiansen (1976) reports that $a(t)$ avoids the non-uniqueness of the solution. Bogomolny (1985) minimizes its influence on the accuracy but ends up with the paradoxical conclusion: the higher the distance between the actual contour and the singularities line, the better the approximation. Pozrikidis (2000) examines in which circumstances the constant must be retained or not; that depends on the existence of a vanishing eigenvalue of the single-layer operator (source distribution on the contour which encloses the whole fluid domain). Goldberg (1995) mentions its dependence on the geometry of the computational domain. By introducing this constant as a normalization factor Mathon and Johnston (1977) or Han and Olson (1987) mention its role in the acceleration of the convergence when the locations of the singularities are a part of the solution (see also Fairweather and Karageorghis, 1998). Smyrlis and Karageorghis (2001) compare the condition numbers (ratio of the smallest to the highest eigenvalues) of the matrices to invert with $a(t) \neq 0$ or $a(t) = 0$. No improvments are observed when the geometry is a disk. Here the considered problem is slightly more complicated since the positions of the singularities vary in time and we may expect strongly irregular singularities line (overturning crest, local jet like flip through,...).

The coefficients G_{ij} of matrix $\mathbf{G}$ are typically the *log* of the distance between (X_j, Y_j) and (x_i, y_i). In practice the position (X_j, Y_j) are determined from the position of the markers (x_j, y_j) and the distance along the normal direction $\vec{n}$ is simply

$$(X, Y) = (x, y) + L_d \vec{n}, \quad \text{with} \quad L_d = \delta \frac{L}{N} \tag{5}$$

In the present application, the Green function is determined so that the presence of impermeable and fixed walls supporting no-flux condition are implicitely accounted for (see Tuck, 1998 or Scolan *et al*, 2007). That means we use a conformal mapping to turn the physical domain into a half plane or sometimes a quarter plane. The Green function is hence expressed in terms of the distance between the markers and the singularities in the transformed plane and NOT in the physical plane. In addition depending on the type of conformal mapping used, the Green function might be the sum of 2 or 4 contributions by mirroring the singularities located in the actual domain with respect to the solid boundaries. In other words the diagonal coefficients G_{ii} do definitely not simplify to $log(L_d)$. It is slightly more complicated however we know that the modulus of the Jacobian of the transformation can be considered as a local ratio of distances in physical and transformed planes.

We are interested in the influence of two parameters: the desingularizing distance L_d and the additive constant $a(t)$. In the present problem we do not have any exact solutions except asymptotically if the fluid domain remains unchanged in time, that is to say when the problem is linearized. Therefore we have only two ways of controlling the accuracy and stability: 1) the conservation of mass and energy, and 2) the conditioning of the matrix $\mathbf{G}$.

2.1) Conservation of mass and energy

In Potential Theory and without artificial dissipation, we must check that both mass and energy are conserved at least at the order of the chosen time marching scheme. If we note $W(t)$ the volume of fluid at any time, we must verify the mass conservation law

$$W(t) = \int_{\Omega(t)} dv = \int_{FS(t)} y\vec{y} \cdot \vec{n} d\ell = \text{constant}, \quad \frac{dW}{dt} = \int_{FS(t)} \phi_{,n} d\ell = \sum_{j=1}^{N} q_j \int_{FS(t)} G_{j,n} d\ell = 0 \tag{6}$$

Ideally $\frac{dW}{dt} = 0$ by virtue of Gauss theorem and by reminding that walls are fixed in time. By noting E the total energy (kinetic and potential) the energy conservation law reads

$$\frac{dE}{dt} = 0, \quad E(t) = \int_{\Omega(t)} \left[\frac{1}{2} \rho \vec{\nabla}^2 \phi + g\rho(y - h) \right] dv = \frac{1}{2} \rho \int_{FS(t)} \left[\phi \vec{\nabla} \phi + g(y - h)^2 \vec{y} \right] \vec{n} d\ell \tag{7}$$

In both energy expressions, the integration is performed over the free surface $FS(t)$ only. These numerical verifications reveal the limitations of the desingularized techniques at least for free surface flows. One reason

is the dependency of conditions (6) and (7) on the desingularizing distance L_d. In addition no attempts are done here to smooth out saw-tooth instabilities or to regrid the free surface. Surprisingly these numerical problems do not occur. Hence in order to improve both accuracy and stability we must identify the best choices of L_d. To this end, an aternative would be to solve additional equations, but then MFS would also lose all its attractivity.

2.2) Roles of L_d and $a(t)$

The way to proceed is rather heuristic as many MFS promoters did so far. However the litterature is rather poor regarding the free surface motion. Cao *et al* (1991) or Tuck (1998) do not enter the numerical analysis of MFS, they only justify their choice of L_d by having stable enough results. They did not even consider the role of $a(t)$. Here we aim at examining the influence of L_d and $a(t)$ on both stability and invariants properties. However we also end up with some paradoxical conclusions.

At first we reformulate the linear system (4) by introducing $a(t)$ with the following form

$$a(t) = b\sum_{j=1}^{N} q_j(t) \tag{8}$$

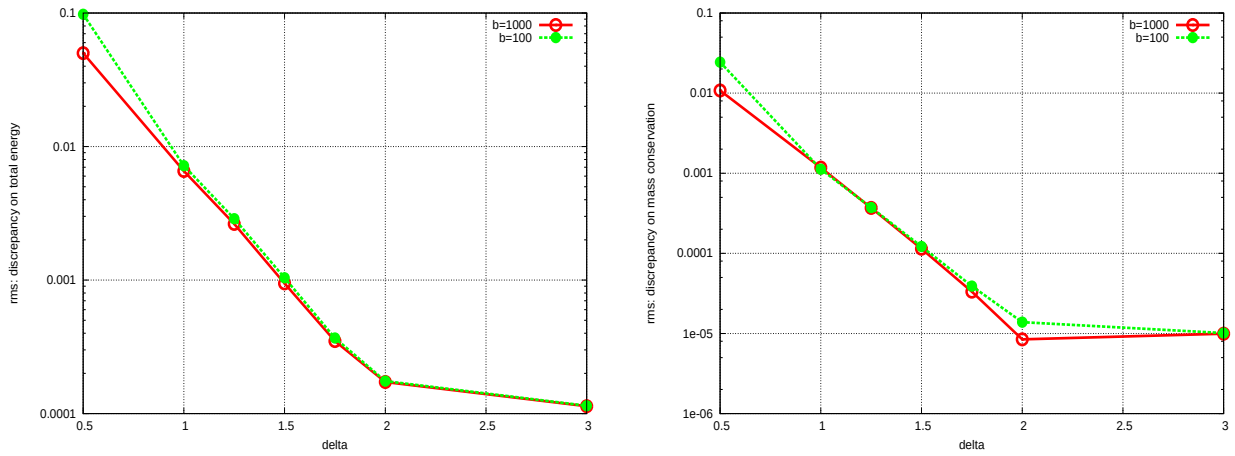
where b does not depend on time (and obviously nor on space). In practice we invert the matrix $\mathbf{G} + b\mathbf{1}$ instead of $\mathbf{G}$. All coefficients of matrix $\mathbf{1}$ is unity. It is worth noting that we can ideally check the following identity

$$(\mathbf{G} + b\mathbf{1})^{-1} = \mathbf{G}^{-1}, \quad \text{if} \quad \sum_{i=1}^{N} G_{ij}^{-1} = 0, \quad \forall j = 1, .., N \tag{9}$$

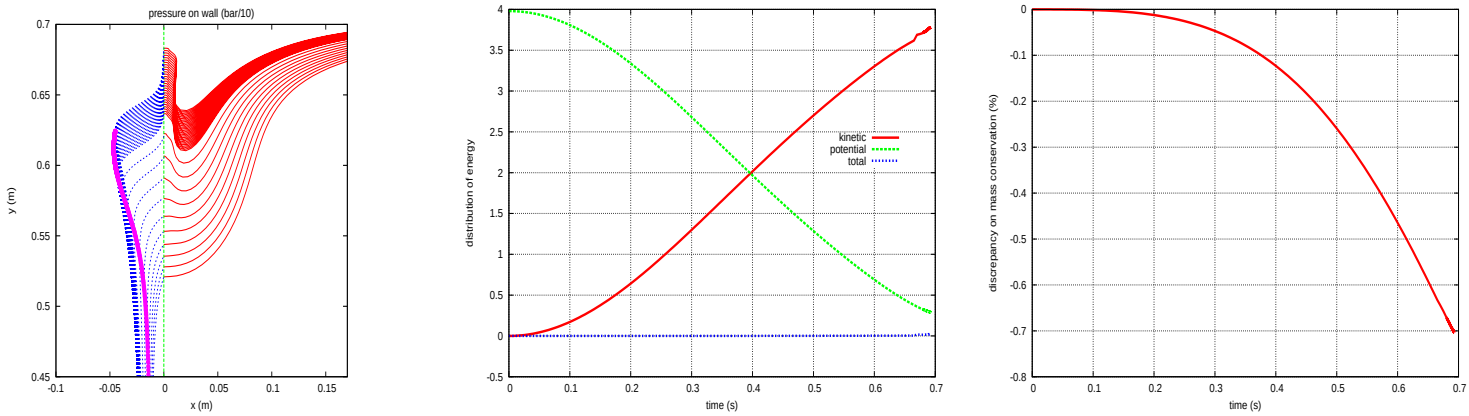
i.e. the summation of the coefficients of $\mathbf{G}$ by column is zero. Paradoxically if condition (9) is fulfilled, it would show that the problem is highly ill-conditioned since this condition precisely means that lines of matrix $\mathbf{G}$ are not linearly independent.

Instead of quantifying the conditionning of the matrix $\mathbf{G} + b\mathbf{1}$ by computing its eigenvalues, we prefer an alternative which perfectly fits to the chosen Gauss algorithm. We systematically evaluate the number of exact digits during the resolution. To this end, we solve the linear system with an additional right hand side made of the summation per line of the coefficients of the matrix to be inverted; the solution must be unity uniformly. The discrepancy with unity should be of order of the precision of the used processor, here 10^{-15} and the exponent (here 15) gives the number of exact digits. Depending on the considered application, the accuracy may vary. For all performed computations, the critical accuracy is set to 10^{-6}, above the code stops.

As first results, we consider a weakly nonlinear standing wave. The time variations of the volume dW/dt and dE/dt are analyzed. Their discrepancies to zero are computed and averaged over long simulations. We start from an initial free surface deformation defined by the first mode of a rectangular basin. The length is $L = 4m$, the mean water depth is $h = 0.75m$ and the amplitude of the mode is $A = 0.2m$. The number of markers is $N = 80$ and remains unchanged during the simulation. They are initially set regularly along the free surface line. Afterwards they are "free to move as they want". The variation of the corresponding RMSs are plotted in the figure below in terms of δ (see eq. 5) and b.



It is difficult to define precisely a variation with b as soon as b is high enough. However the influence of the desingularizing distance is evident. For the present application we can recommend "good" choices of b and δ. Those coefficients are not the same when high distorsion of the free surface is simulated. As an example we catch flip through phenomenon starting from a well chosen initial free surface deformation. A hyperbolic tangent is classicaly used with the highest slope in the middle of the tank. When modelling flip through, we mainly aim at reaching the highest loads and the highest kinematics locally along the vertical wall. According to Scolan (2009) the peak of acceleration preceeds the peak of pressure. Cooker (2009) shows from a local asymptotic analysis an inverse chronology. Longuet-Higgins (2001) suggests otherwise that these two maxima are concomitant. Figures below show (on the left) the pressure profile on the wall and the corresponding free surface profiles. The two figures on the right show the time variations of mass and energy components (kinetic, potential and total).



It is clear that large errors can be reached (still less than 1%). The reasons are due to the adaptive desingularizing distance to the small local curvature radius as the jet develops and that is obviously unavoidable. The question is to know whether or not that is acceptable.

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