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Green function with dissipation and side wall effect in wave tanks

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The Green function with viscous dissipation associated with a pulsating source in a wave tank is considered within the theory of visco-potential flow presented in Chen & Dias (2010). The so-called Tank Green function (TGF) with the side-wall effect in wave tanks is formulated as a formal sum of open-sea Green functions representing the infinite images between two parallel side walls of the source in the tank. The introduction of fluid viscosity gives rise to a decay factor in the TGF which is not singular any more at the wavenumbers associated with the transversal resonances. Furthermore, the new integrals representing the far-field waves obtained by analyzing the truncated infinite series of the wave component of the open-sea Green function are expressed as the analytical formulations involving the complementary error function.

1. Tank Green function with dissipation

A Cartesian coordinate system is defined by placing the (x, y)-plane coincided with the undisturbed free surface and the z-axis oriented positively upward. The x-axis is coincident with the center plane of the tank whose width is denoted by b. Under the assumption of fairly perfect fluid (Chen & Dias, 2010), the TGF $G(P, Q)$ representing the velocity potential at a field point $P(x, y, z)$ in the wave tank due to a pulsating source of unit strength located at the point $Q(x', y', z')$, satisfies the following set of equations :

$$\nabla^2 G(P,Q) = \delta(P-Q) \qquad P \subset D \tag{1a}$$

$$G_z - \bar{k}G - i4\alpha G_{zz} = 0 \qquad z = 0 \tag{1b}$$

$$G_z = 0 \qquad z = -h \tag{1c}$$

$$G_y = 0 \qquad y = \pm b/2 \tag{1d}$$

where $\delta(\cdot)$ is the Dirac function and the parameters $(\bar{k}, \alpha)$ are defined as

$$\bar{k} = \omega^2/g \quad \text{and} \quad \alpha = \mu\omega/(\rho g)$$

with ω the frequency of pulsating source, g the acceleration of the gravity, μ the fluid viscosity and ρ the fluid density. The solution of (1) can be obtained by considering an infinite number of images of the source between two parallel side-walls, that is :

$$G(P,Q) = \sum_{n=-\infty}^{\infty} G_n^0(P, Q_n) \tag{2}$$

where G_n^0 is the open-sea Green function with viscous dissipation satisfying the first three equations in (1), representing the velocity potential at the point P due to the nth image of the source located at $Q_n(x', y'_n, z')$ with the coordinate y_n defined by :

$$y'_n = (-1)^n y' + nb \tag{3}$$

The direct computation of the infinite series (2) is slowly convergent. The tank Green function can be regrouped into two parts which has been proved to be more computationally efficient :

$$G = G^F + G^H \tag{4}$$

with G^F a finite series :

$$G^F = \sum_{n=-2N-1}^{2N+1} G_n^0(P, Q_n) \tag{5}$$

and the remaining by the truncated infinite series

$$G^H = \sum_{n=N+1}^{\infty} (G_{2n}^0 + G_{2n+1}^0 + G_{-2n}^0 + G_{-2n-1}^0) \tag{6}$$

which represents the contribution of the source images far from the field point.

In the following, the new open-sea Green function G_n^0 with viscous effect is developed and the transformation of G^H into two single integrals whose numerical evaluation method is analyzed.

2. Open-sea Green function with viscosity

The general solution of the open-sea Green function which satisfies the first three equation of (1) can be written as

$$4\pi G^0(P,Q) = -1/|PQ| - 1/|PQ_2| - H(r,z,z') \tag{7}$$

where $Q_2(x',y',-z'-2h)$ is the symmetrical point of $Q(x',y',z')$ with respect to the sea bed $z=-h$. The free-surface term $H(r,z,z')$ under the cylindrical coordinates with $r=\sqrt{(x-x')^2+(y-y')^2}$ should satisfy

$$(\partial_{rr}+r^{-1}\partial_r+\partial_{zz})H = 0 \qquad (r,z)\subset D \tag{8a}$$
$$(\partial_z-\bar{k}-i4\alpha\partial_{zz})H = (\partial_z-\bar{k}-i4\alpha\partial_{zz})(1/|PQ|+1/|PQ_2|) \qquad z=0 \tag{8b}$$
$$\partial_z H = 0 \qquad z=-h \tag{8c}$$

By the standard Fourier-Hankel transform of (8), we get the following expression:

$$H = 2\int_0^\infty \frac{k+\bar{k}+i4\alpha k^2}{k\sinh(kh)-(\bar{k}+i4\alpha k^2)\cosh(kh)}\cosh k(z+h)\cosh k(z'+h)\,\mathrm{J}_0(kr)\,dk \tag{9}$$

with $\mathrm{J}_0(\cdot)$ the zeroth-order Bessel function of the first kind defined in Abramowitz & Stegun (1967).

According to the perturbation method, by omitting the terms of order $O(\alpha^2)$ or higher, and using the same technique as in John (1950) to evaluate H, the open-sea Green function with viscous effect can be expressed as

$$G^0 = \frac{-ik_0}{2k_0h+\sinh(2k_0h)}\cosh k_0(z+h)\cosh k_0(z'+h)\,\mathrm{H}_0(k_0r+i4\alpha k_{0I}r)$$
$$-\sum_{n=1}^{\infty}\frac{2k_n/\pi}{2k_nh+\sin(2k_nh)}\cos k_n(z+h)\cos k_n(z'+h)\,\mathrm{K}_0(k_nr+i4\alpha k_{nI}r) \tag{10}$$

in which $\mathrm{H}_0(\cdot)$ and $\mathrm{K}_0(\cdot)$ are the zeroth-order Hankel function of the first kind and the zeroth-order modified Bessel function of the second kind defined in Abramowitz & Stegun (1967), the wavenumbers k_0 and k_n for $n\geq 1$ are the real roots of the classical dispersion equations :

$$k_0\tanh(k_0h) = \bar{k} \quad \text{and} \quad k_n\tan(k_nh) = -\bar{k} \tag{11}$$

respectively, while the values of k_{0I} and k_{nI} for $n\geq 1$ are defined by :

$$k_{0I} = 2k_0^2\cosh^2(k_0h)/[2k_0h+\sinh(2k_0h)] \quad \text{and} \quad k_{nI} = 2k_n^2\cos^2(k_nh)/[2k_nh+\sin(2k_nh)] \tag{12}$$

which show the explicit relation between (k_{0I},k_{nI}) and (k_0,k_n) for $n\geq 1$, respectively. In (10), the first term on the right hand side involving H_0 is often called the wave component while the second term with the sum involving K_0 is evanescent.

3. Asymptotic part of TGF and integral representations

Assuming that the lowest number $2N+1$ in (2) is large enough to neglect the evanescent part of the open-sea Green function (10), Chen (1994) points out that the asymptotic part G^H given by (6) of the TGF can be rewritten as the sum including two infinite single integrals :

$$G^H = \frac{-ik_0\cosh k_0(z+h)\cosh k_0(z'+h)}{2k_0h+\sinh(2k_0h)}\sum_{n=1}^{4}c(Y_n,B)\Big[I_1(Y_n,B)+id(X)I_2(Y_n,B)/B\Big] \tag{13}$$

in which

$$c(Y_n,B) = e^{i(2BY_n-\pi/4)}/(\pi B^{1/2}) \tag{14}$$

and

$$d(X) = \sum_{m=0}^{\infty}\epsilon_m\,\mathrm{J}_{2m}(X)(16m^2-1)/8 = (4X^2-1)/8 \tag{15}$$

where $\mathrm{J}_{2m}(\cdot)$ are the $2m$th-order Bessel functions of the first kind, $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$. In (13), (14) and (15), we have used :

$$\begin{aligned}
B &= (k_0 + i4\alpha k_{0I})b \\
X &= (k_0 + i4\alpha k_{0I})(x - x') \\
Y_1 &= N + 1 - (y - y')/(2b) \\
Y_2 &= N + 1 + (y - y')/(2b) \\
Y_3 &= N + 3/2 - (y + y')/(2b) \\
Y_4 &= N + 3/2 + (y + y')/(2b)
\end{aligned}$$

Finally, the two infinite integrals are given by :

$$I_1(Y_n, B) = \sqrt{\pi} \sum_{m=0}^{\infty} e^{i2mB}/(m + Y_n)^{1/2} = \int_0^{\infty} e^{-Y_n t} t^{-1/2}/(1 - e^{i2B-t})\, dt \tag{16a}$$

$$I_2(Y_n, B) = \sqrt{\pi}/2 \sum_{m=0}^{\infty} e^{i2mB}/(m + Y_n)^{3/2} = \int_0^{\infty} e^{-Y_n t} t^{1/2}/(1 - e^{i2B-t})\, dt \tag{16b}$$

Using the Taylor development of e^{-t} in the denominator of two infinite integrals (16), the single integral I_1 defined by (16a) is approximated in Chen & Xia (2005) by :

$$\tilde{I}_1 = \pi e^A \,\mathrm{erfc}(\sqrt{A})/\sqrt{(1 - e^{i2B})e^{i2B}} \tag{17a}$$

with $A = Y_n(e^{-i2B} - 1)$ and $\mathrm{erfc}(\cdot)$ the complementary error function.

In the same way, the single integral I_2 defined by (16b) can be approximated by :

$$\tilde{I}_2 = [\sqrt{\pi/A} - \pi e^A \,\mathrm{erfc}(\sqrt{A})]\sqrt{(1 - e^{i2B})/e^{i6B}} \tag{17b}$$

To note that the single integral I_2 (16b) is always convergent since its major value $\tilde{I}_2$ (17b) is finite regardless of the value of B. The function e^{i2B} present in (17a) and (17b) has the property :

$$|e^{i2B}| = |e^{i2k_0 b - 8\alpha k_{0I} b}| < 1 \tag{18}$$

since $8\alpha k_{0I} b > 0$ from (12), the value of $\tilde{I}_1$ (17a) is always finite and the original integral I_1 (16a) convergent. In the case without taking account of viscous effect by putting $\alpha = 0$, the value $\tilde{I}_1$ (17a) tends to infinity for a set of discrete values of $B = \kappa\pi$ with $\kappa = 1, 2, \cdots$, as shown in Chen (1994), which are associated with the resonance modes of transversal waves between two vertical walls.

4. Discussions and concluding remarks

New formulations of the open-sea Green function with viscous effect in water of finite depth are developed within the linear theory of visco-potential flow in a fairly perfect fluid (Chen & Dias, 2010). Unlike the inviscid potential flow, the far-field behavior of the velocity potential represented by the Hankel function in (10) is :

$$\mathrm{H}_0(k_0 r + i4\alpha k_{0I} r) \approx e^{-4\alpha k_{0I} r}\sqrt{2/[\pi(k_0 + i4\alpha k_{0I})r]}\exp(ik_0 r - i\pi/4) \tag{19}$$

for $r \to \infty$. The decay factor $e^{-4\alpha k_{0I} r}$ represents the dissipation effect of fluid viscosity which is absent in the classical inviscid potential flow. As illustrated on Figure 1, the magnitude of the complex Hankel function depicted by its real and imaginary parts decreases as $O(1/\sqrt{k_0 r})$ without viscous dissipation ($\alpha = 0$) while it decays much faster in the order of $O(e^{-4\alpha k_{0I} r}/\sqrt{k_0 r})$ with viscous dissipation ($\alpha \neq 0$). The Green function with dissipation effect must be particularly interesting in the solution of wave diffraction and radiation around one or several floating bodies.

The construction of the tank Green function by an infinite series of open-sea Green function to evaluate the side wall effects in wave tanks is summarized in the paper. The asymptotic part G^H defined by the truncated infinite series (6) is transformed into a sum (13) including two single integrals (16). The new analysis of the single integrals provides their analytical formulations (17a) and (17b) which represent the major part of the integrals. The remaining value corresponding to the difference between the original single integral and the analytical formula can be evaluated numerically and approximated by polynomials. The analysis of the single integral (16a) and its analytical expression (17a) shows that it is finite due to the decay factor associated with the dissipation, unlike the classical TGF which is singular at the wavenumbers associated with the resonance of transversal waves between two parallel side walls.

Another important source of dissipation is due to the contact of waves against the side walls along which special dampers might be installed to reduce the reflection from the walls. The idea of partial reflection from the walls can be introduced in the infinite series to represent the deficiency of wave energy. If a constant partial reflection coefficient is applied in the series, it is mathematically equivalent to introduce an additional decay factor. The same development as above can be carried out in a much easier way.

A number of studies have been performed in order to evaluate the side-wall effect in wave tanks. Most of them, such as Eatock Taylor & Hung (1985), Yeung & Sphaier (1989), Kashiwagi (1991) or McIver (1993), give analytical or semi-analytical solutions using eigenfunction expansion or multipoles with limitation to bodies of simple geometry such as vertical cylinders placed in the center of wave tank. Some few studies, like Linton (1993), Chen (1994), Xia (2002) or Chen & Xia (2005), deal with the formulation and numerical computations of TGF and the first-order and second-order solutions using boundary element method. The inviscid potential flow free of any dissipation is modeled in these previous studies. The present study on the TGF including the dissipation effect is expected to give important insight on the realistic effect of side walls in wave tanks and to be able to provide closer results to the measurement of model tests.

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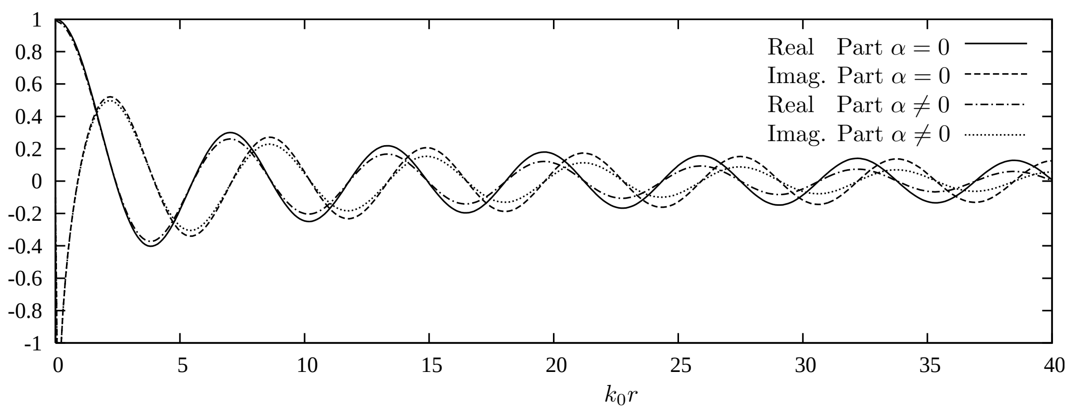

Figure 1: Free-surface waves with and without viscous dissipation