Wave focusing over submerged elliptical topography

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1. Introduction

The idea of focusing of surface waves by underwater lenses was first proposed by Mehlum & Stamnes (1978). The basic concept is rather simple: oblique waves are refracted by changes in depth and so as a wave passes from a depth h_1 to a smaller depth h_2, say, the refractive index n determined by $n = k_2/k_1 > 1$ allows oblique waves to straighten out, where k_1 and k_2 are the wavenumbers for travelling waves determined by the linear dispersion relation $K \equiv \omega^2/g = k_i \tanh k_i h_i$, $i = 1, 2$. Mehlum & Stamnes (1978) and subsequent later work by these authors used this idea to consider the focusing of surface waves by lenses which comprised horizontal underwater plates forming a 'Fresnel lens' (the type used in lighthouses and overhead projectors for example) in plan form, although a conventional convex lens would refract waves equally well. Thus incoming waves passing across the lens are transformed into a circular wave which converges at the focal point of the lens. See, for example, Stamnes *et al.* (1983), Murashinge & Kinoshita (1992) and references therein.

Kudo *et al.* (1989) used similar ideas, employing a submerged horizontal plate in the shape of a lens to refract waves. In plan form ther lens had an elliptical-arc leading edge and a circular-arc trailing edge. Here, the authors were exploiting ray theoretical idea that incoming parallel rays entering an elliptical domain with refractive index $n = 1/\epsilon$ where ϵ is the ellipticity are exactly focussed on the far focal point P of the ellipse (see §2).

Here we consider focusing of waves by elliptical topographic features. Specifically, we examine the refraction of waves in otherwise constant depth h_1 incident on an elliptical mound, with a plateau at depth $h_2 < h_1$. According to ray theory high frequency surface waves will be refracted by the change in depth and focus above the far focal point of the elliptical plateau. Of course, the change in depth could be effected by having waves pass across a submerged elliptical plate. Such a problem was considered by Zhang & Williams (1996) although evidently they are unaware of the ray theory result of exact focusing.

In order to examine the focusing by elliptical topography we consider two separate approaches. The first, detailed in this abstract, is based on the modified mild-slope equations (Chamblerlain & Porter (1995)) which represent the three-dimensional fluid motion by two-dimensional depth-averaged equations based on the assumption that the gradient of the bed is small compared to the wavelength. In this problem we therefore consider a sea-mount which rises gradually and smoothly from the open depth h_1 onto the elliptical plateau of depth h_2.

In a second approach we take a vertically-sided submerged elliptical mount and this allows us to use exact three dimensional linear wave theory to consider the diffraction of incident waves, as we can now employ separation solutions in elliptical coordinates. Results of this investigation will be presented at the workshop.

2. Motivation: Ray Theory

Consider an elliptical domain with refractive index $n > 1$ and major axis $2a$, minor axis $2b$. Then the eccentricity is defined as

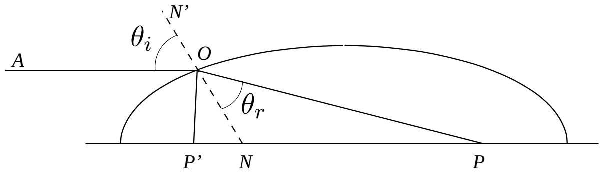

Figure 1: Focusing of an incident ray on the far focal point of an ellipse

$\epsilon = \sqrt{1 - b^2/a^2}$ and the focal points P and P' lie at $\pm a\epsilon$. See figure 1. According to ray theory, an incident AO ray parallel to the major axis makes an angle θ_i with the normal NN' to the boundary at O. The ray proceeds from O at an angle θ_r with respect to NN' where Snell's law relates θ_i to θ_r by $\sin\theta_i/\sin\theta_r = n$. The ray intercepts the major axis at P. P' is a point at which the ray PO would be *reflected* at O by the boundary back onto the axis. Then $\angle ONP = \pi - \theta_i$ and by the sine rule $OP = nNP$. Also $\angle NOP' = \theta_r$ whilst $\angle ONP' = \theta_i$ and now the sine rule gives us $OP' = nNP'$. Summing these two results gives $POP' = nPNP'$ and, to be independent of O, we must have P and P' at the focal points when we get $2a = n2a\epsilon$ or $n = 1/\epsilon$.

When considering water waves in the short wavelength limit, a wave approaching the point O sees a change in depth along a curve which is locally straight. Insisting that there is no change in the component of the wavenumber parallel to this boundary, gives $k_1 \sin\theta_i = k_2 \sin\theta_r$ where k_1 and k_2 are wavenumbers of propagating waves in depths h_1 and h_2 outside and inside the elliptical boundary. Thus an approximate relation, based on short-wavelength theory, for focusing waves requires

$$n = k_2/k_1 = 1/\epsilon \tag{1}$$

be satisfied. The practical implications of this relation is that for a given ratio $h_2/h_1 < 1$ and incident wavenumber k_1h_1, (1) above determines ϵ required for focusing.

3. Formulation of solution

The sea-bed is given by $z = -h(x,y)$ where $h(x,y)$ is a continuous function and is such that $h(x,y) = h_1$, a constant, outside a finite domain $(x,y) \in D$ is assumed arbitrary for the moment.

Using linearised water wave theory a velocity potential is given by $\Re\{\Phi(x,y,z)\mathrm{e}^{-\mathrm{i}\omega t}\}$ where ω is the assumed angular frequency of motion. Then $\Phi(x,y,z)$ satisfies

$$(\nabla^2 + \partial_{zz})\Phi = 0, \quad -h(x,y) < z < 0 \tag{2}$$

where $\nabla = (\partial_x, \partial_y)$,

$$\Phi_z - \nu\Phi = 0, \quad \text{on } z = 0 \tag{3}$$

where $\nu = \omega^2/g$, g is gravitational acceleration and

$$\Phi_z + \nabla h.\nabla\Phi = 0, \qquad \text{on } z = -h(x,y) \tag{4}$$

which reduces to $\Phi_z = 0$, on $z = -h_1$ for $(x,y) \notin D$.

An incident wave of unit amplitude progressing at an angle β to the x axis from infinity over the flat bed is given by the potential

$$\Phi_{inc}(x,y,z) = \mathrm{e}^{\mathrm{i}k_1(x\cos\beta + y\sin\beta)} f(k_1h_1, k_1z)$$

where

$$f(kh, kz) = \frac{\cosh(kh + kz)}{\cosh kh} \tag{5}$$

and $k = k_1$ is the real positive root corresponding to $h = h_1$ of

$$k \tanh kh = \nu. \tag{6}$$

The total potential is written as $\Phi = \Phi_{inc} + \Phi_{sc}$ where Φ_{sc} is the scattered wave potential resulting from the interaction of the incident wave with the undulating part of the topography in $(x,y) \in D$. At infinity Φ_{sc} satisfies the radiation condition,

$$\Phi_{sc} \sim A(\theta;\beta)\sqrt{\frac{2}{\pi k_1 r}}\,\mathrm{e}^{\mathrm{i}(k_1 r - \pi/4)} f(k_1h_1, k_1z),$$

as $k_1 r \to \infty$ where $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$ and $A(\theta;\beta)$ is the diffraction coefficient.

An approximate solution is sought by the mild-slope method. That is, assuming the depth-dependence assigned to propagating modes over a locally-flat bed

$$\Phi(x, y, z) \approx \phi(x, y) f(kh, kz) \qquad (7)$$

in which $k(h(x, y))$ denotes the positive, real root of (6) where the depth is $h(x, y)$. Chamberlain & Porter (1995) implemented the approximation (7) by using a variational principle which replaces (2), (3) and (4) with the by the single modified mild-slope equation

$$\left.\begin{aligned} \nabla.(u_0\nabla\phi) + v_0\phi = 0, \quad (x, y) \in \mathbb{R}^2, \\ v_0 = k^2 u_0 + u_1\nabla^2 h + u_2(\nabla h)^2. \end{aligned}\right\} \quad (8)$$

where

$$u_0 = \operatorname{sech}^2 kh(2kh + \sinh 2kh)/4k. \qquad (9)$$

The coefficients u_1 and u_2, need not be given explicitly here; we remark that if u_1, u_2 are set to zero, (8) reduces to the simpler mild-slope equation.

The use of (7) implies that $\phi(x, y) = \phi_{inc}(x, y) + \phi_{sc}(x, y)$ where $\phi_{inc}(x, y) = \mathrm{e}^{\mathrm{i}k_1(x\cos\beta + y\sin\beta)}$ and

$$\phi_{sc}(x, y) \sim A(\theta; \beta)\sqrt{\frac{2}{\pi k_1 r}}\,\mathrm{e}^{\mathrm{i}(k_1 r - \pi/4)}, \quad (10)$$

as $k_1 r \to \infty$.

Equation (8) is transformed into its canonical form, by writing

$$\phi(x, y) = \{u_0(h_1)/u_0(h(x, y))\}^{1/2}\psi(x, y). \quad (11)$$

Then ψ satisfies

$$\nabla^2\psi + \kappa(x, y)\psi = 0, \quad (x, y) \in \mathbb{R}^2 \qquad (12)$$

where

$$\kappa(x, y) = k^2 + A\nabla^2 h + B(\nabla h)^2 \qquad (13)$$

and, with the abbreviation $K = 2kh$,

$$\begin{aligned} A &= -2k/(K + \sinh K), \\ B &= k^2\{K^4 + 4K^3\sinh K + 3K^2(2\cosh^2 K + 1) \\ &\quad + 18K\sinh K + 3\sinh^2 K(2\cosh K + 5)\}/ \\ &\quad \{3(K + \sinh K)^4\}, \end{aligned}$$

include the functions u_1 and u_2 appearing in (8). Finally we mimic the decomposition of ϕ writing $\psi(x, y) = \psi_{inc}(x, y; \beta) + \psi_{sc}(x, y)$.

We reformulate the problem for ψ as an integral equation, making using of the fact that, for $(x, y) \notin D$, (12) reduces to the Helmholtz equation $(\nabla^2 + k_1^2)\psi = 0$.

Thus, we introduce the Green's function $G(x, y; x', y') \equiv G(\mathbf{x}; \mathbf{x}')$ defined by

$$(\nabla^2 + k_1^2)G = \delta(x - x')\delta(y - y'), \qquad (14)$$

for $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ and given by

$$G(\mathbf{x}; \mathbf{x}') = -\tfrac{1}{4}\mathrm{i}H_0(k_1\rho) \qquad (15)$$

where $H_0(x)$ is the Hankel function of the first kind and $\rho = |\mathbf{x} - \mathbf{x}'|$. Thus $G \sim (1/2\pi)\ln(k_1\rho)$ as $k_1\rho \to 0$ whilst $G \sim -\frac{1}{4}\mathrm{i}\sqrt{2/\pi k_1\rho}\,\mathrm{e}^{\mathrm{i}(k_1\rho - \pi/4)}$ as $k_1\rho \to \infty$.

Applying Green's Identity to $\psi(\mathbf{x})$ and $G(\mathbf{x}; \mathbf{x}')$ over $\mathbf{x} \in \mathbb{R}^2$ gives the second kind integral equation

$$\begin{aligned} \psi(\mathbf{x}') + \iint_D [\kappa(\mathbf{x}) - k_1^2]G(\mathbf{x}; \mathbf{x}')\psi(\mathbf{x})\mathrm{d}\mathbf{x} \\ = \psi_{inc}(\mathbf{x}'; \beta), \quad \mathbf{x}' \in \mathbb{R}^2. \quad (16) \end{aligned}$$

The equation (16), when restricted to $\mathbf{x} \in D$, serves as an integral equation for the unknown reduced potential $\psi(\mathbf{x})$ on D, whilst ψ is determined elsewhere by applying (16) to points $\mathbf{x} \notin D$.

It is not difficult to express the diffraction coefficient as

$$A(\theta; \beta) = \tfrac{1}{4}\mathrm{i}\iint_D [\kappa(\mathbf{x}) - k_1^2]\psi(\mathbf{x})\overline{\psi_{inc}(\mathbf{x}; \theta)}\mathrm{d}\mathbf{x}.$$

The free surface elevation due to an incident wave of unit amplitude is given by $\eta(x, y) = \Phi(x, y, 0)$ and therefore

$$\eta(x, y) = \{u_0(h_1)/u_0(h(x, y))\}^{1/2}\psi(x, y)$$

where u_0 is given by (9).

The integral equation is solved numerically as follows. For simplicity consider a rectangular domain $D = D_{ab}$ given by $-a \le x \le a$ and $-b \le y \le b$ and define an $N \times M$ array of points $\mathbf{x}_{i+(j-1)N} = (x_i, y_j)$ where

$$\left.\begin{aligned} x_i = a - (i - \tfrac{1}{2})\Delta_x, \qquad i = 1, \ldots, N \\ y_j = b - (j - \tfrac{1}{2})\Delta_y, \qquad j = 1, \ldots, M \end{aligned}\right\}$$

with $\Delta_x = 2a/N$ and $\Delta_y = 2b/M$. The integral in (16) is approximated using the rectangle midpoint rule with evaluations of the integrand made at $\mathbf{x} = \mathbf{x}_i$, $i = 1, \ldots, NM$

$$\psi(\mathbf{x}') + \Delta_x \Delta_y \sum_{i=1}^{NM} [\kappa(\mathbf{x}_i) - k_1^2] G(\mathbf{x}_i; \mathbf{x}') \psi(\mathbf{x}_i) = \psi_{inc}(\mathbf{x}'; \beta), \tag{17}$$

collocating at points $\mathbf{x}' = \mathbf{x}_j$, $j = 1, \ldots, NM$ resulting in the system

$$\psi(\mathbf{x}_j) + \Delta_x \Delta_y \sum_{i=1}^{NM} [\kappa(\mathbf{x}_i) - k_1^2] G_{ij} \psi(\mathbf{x}_i) = \psi_{inc}(\mathbf{x}_j; \beta), \quad j = 1, \ldots, NM$$

for unknowns $\psi(\mathbf{x}_j)$, where $G_{ij} = G(\mathbf{x}_i; \mathbf{x}_j)$, $i \neq j$, and

$$G_{jj} = \widetilde{G}(\mathbf{x}_j; \mathbf{x}_j) + R_j, \qquad i = j,$$

and $\widetilde{G}(\mathbf{x}; \mathbf{x}') = -\frac{1}{4}\mathrm{i}H_0(k_1\rho) - (1/2\pi)\ln(k_1\rho)$ is regular as $\rho \to 0$ giving $\widetilde{G}(\mathbf{x}_j; \mathbf{x}_j) = (\gamma - \log 2)/(2\pi)$ where $\gamma = 0.577...$ is Euler's constant whilst

$$R_j = [\ln(\Delta_x^2 + \Delta_y^2) + (\Delta_y/\Delta_x)\tan^{-1}(\Delta_x/\Delta_y) + (\Delta_x/\Delta_y)\tan^{-1}(\Delta_y/\Delta_x) - 3]/4\pi$$

is the result of integrating $(1/2\pi)\ln(k_1\rho)$ exactly over the rectangle Δ_x by Δ_y.

4. Results

Examples of focusing are shown in figs 2(a),(b). In both cases $h_2/h_1 = \frac{1}{2}$ (a modest size to miminise diffraction effects) and $a/h_1 = 40$. The shape $h(x, y)$ of the topography is a function of an elliptical coordinate u only and consists of a smooth cosine transition between the values h_1 to h_2 over half the range of u. Then in fig 3(a), we have $k_1h_1 = 1$ and this determines $b/h_1 = 26$ from (1). In fig 2(b) we have $k_1h_1 = \frac{1}{2}$ from which $b/h_1 = 27$. In both cases, clear signs of focusing close to the predicted value of $x/h_1 = 27$ can be seen. In fig 3(b), signs of phase-related interference can be seen. These features and others will be discussed at the workshop.

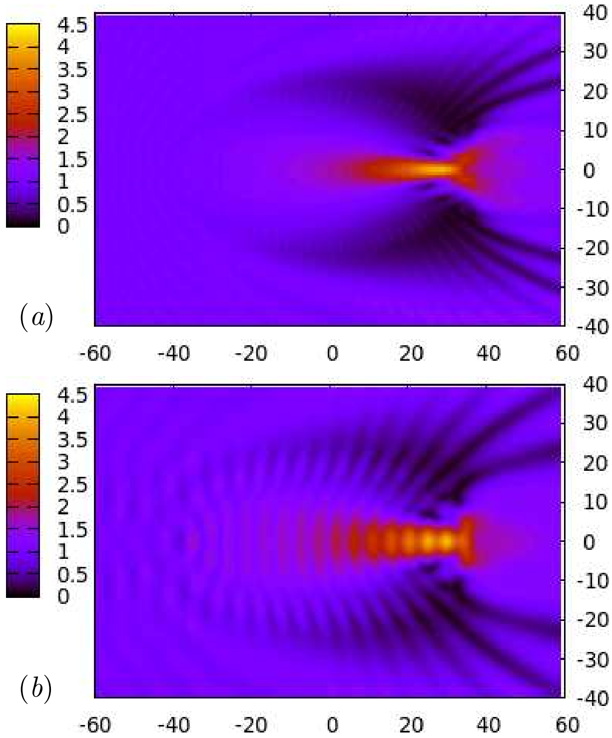

Figure 2: (a) $k_1h_1 = 1$ and (b) $k_1h_1 = \frac{1}{2}$

5. References

1. Kudo, K., Tsuzuku, T., Iwai, K. & Akiyama, Y., 1988. Wave focusing by a submerged plate. OCEANS '88. 'A Partnership of Marine Interests'. Proceedings. pp.1061–1066.
2. Mehlum, E. & Stamnes, J., 1978, On the focusing of ocean swells and its significance in power production. Cent.Inst. for Indust. Res., Blindern, Oslo, SI Rep. 77 01 38.
3. Stamnes, J., Lovhaugen, O., Spjelkavik, B., Mei, C.C., Lo, E. & Yue, D.K.P., 1983, Nonlinear Focusing of Surface Waves by a Lens - Theory and Experiment *J. Fluid Mech.* **135** 71–94.
4. Chamblerlain, P.G. & Porter, D., 1995, The modified mild-slope equation. *J. Fluid Mech.* **291** 393–407.
5. Zhang, S. & Williams, A.N., 1996, Wave scattering by submerged elliptical disk. *J. Waterway, Port, Coastal Ocean Engng.* **122**, 38–45.