Non-uniqueness in the plane problem of steady forward motion of bodies

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1. Introduction

In the note we consider the classical linear problem of ship waves, which appears in the framework of the surface wave theory and describes forward motion of rigid bodies with a constant speed U in an unbounded heavy fluid having a free surface. The fluid is assumed to be ideal and incompressible; its motion is steady-state and irrotational. The corresponding boundary value problem is often referred to as the Neumann–Kelvin problem; it was studied by many authors (see [1] and references therein). We shall consider the two-dimensional statement and the case when the contours of ships are totally submerged.

It is known (see [1, 8]) that the problem is uniquely solvable for all values of U except a finite (possibly empty) set of values. The method of proof gives no information on the exceptional values and the question of their existence was open for many years (it should be mentioned that [8] provide some bounds; also in [3] uniqueness is proved for a circular cylinder for all values of U). Examples of non-uniqueness for the case of surface-piercing bodies were found in [2] by the inverse procedure (see [4, 5]). However, for the problem of motion of totally submerged bodies the procedure appears to be ineffective.

In the present work we shall construct examples of non-uniqueness with the help of the uniqueness criteria found recently in [8]. Unlike the inverse procedure the approach [8] allows us to establish existence of non-uniqueness examples for *given* geometries of submerged bodies. Numerical results confirming existence of non-uniqueness examples are given and discussed.

2. Statement of the problem

We study the forward motion of submerged rigid bodies B with wetted surface $S = \partial B$ through an unbounded fluid W that has a free surface F (see fig. 1).

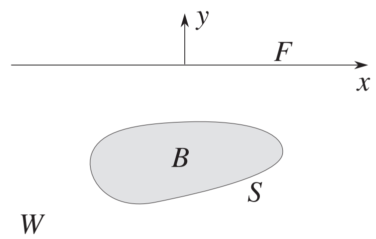

Figure 1: Notations.

A Cartesian coordinate system is attached to the bodies; the x-axis is directed upstream and the y-axis is directed vertically upwards. The motion of the fluid is described by a velocity potential $u(x,y)$ satisfying the following set of conditions (which will be also referred to as problem (A) below):

$$\nabla^2 u = 0 \quad \text{in} \quad W = \mathbb{R}^2_- \setminus \overline{B}, \tag{1}$$

$$\partial_x^2 u + \nu \partial_y u = 0 \quad \text{on} \quad F = \{y = 0\}, \tag{2}$$

$$\partial_{\boldsymbol{n}} u = f \in C(S) \quad \text{on} \quad S \in C^{1,\alpha}, \tag{3}$$

$$\sup_W |\nabla u| < \infty, \qquad \lim_{x \to +\infty} \partial_x u(x,0) = 0, \tag{4}$$

where $\mathbb{R}^2_- = \{y < 0\}$, $\nu = g/U^2$ is the wave number, g is the acceleration due to gravity, $\boldsymbol{n}$ is the unit normal vector on S directed into W, and $f = U\boldsymbol{n}_x$ if impermeability is assumed (here $\boldsymbol{n}_x$ is x-component of the vector). The free surface elevation is equal to $Ug^{-1}\partial_x u\big|_{y=0}$ and the second condition (4) shows that there are no waves at infinity upstream (see [8]). In (3) $C^{1,\alpha}$, $0 < \alpha < 1$, is Hölder's space.

We shall also need an auxiliary problem. Namely, we shall say that u' is a solution to the problem (B) with opposite direction of motion if u' satisfies (1)–(3), the first condition (4) and

$$\lim_{x \to -\infty} \partial_x u'(x,0) = 0.$$

3. Green's function and boundary integral equations

Green's function $G(x,y,\xi,\eta)$ is the potential of a moving source located at a point (ξ,η). The potential satisfies as a function of the first two arguments the conditions (2), (4) (where supremum is taken over $\mathbb{R}^2_-$ with a vicinity of the point (ξ,η) excluded), and the condition $\nabla^2_{x,y}G(x,y,\xi,\eta) = -\delta(x-\xi)\delta(y-\eta)$, where $y,\eta < 0$ and δ is Dirac's delta-function.

We have (see e.g. [1, § 6.3.1])

$$G(z,\zeta) = -\frac{1}{2\pi}\left[\log(\nu|z-\zeta|)+\log(\nu|z-\overline{\zeta}|)\right]$$
$$-\frac{1}{\pi}\int_0^\infty \frac{e^{k(y+\eta)}}{k-\nu}\cos k(x-\xi)\,dk - e^{\nu(y+\eta)}\sin\nu(x-\xi),$$

where $z = x+iy$, $\zeta = \xi+i\eta$, the integral is understood as Cauchy principal value.

Following the usual scheme of potential theory we can seek solutions to the problem (1)–(4) in the form of a single layer potential (see, e.g., [1, § 2.1])

$$u(z) = \big(V\mu\big)(z), \quad z \in W, \tag{5}$$

where

$$\big(V\mu\big)(z) = \int_S \mu(\zeta)\,G(z,\zeta)\,ds_\zeta, \tag{6}$$

and μ is some unknown density belonging to $C(S)$.

The potential (5) satisfies conditions (1), (2), (4) and the condition (3) leads to the boundary integral equation

$$-\mu(z) + \big(T\mu\big)(z) = 2f(z), \quad z \in S, \tag{7}$$

where

$$\big(T\mu\big)(z) = 2\int_S \mu(\zeta)\,\partial_{\boldsymbol{n}(z)}G(z,\zeta)\,ds_\zeta$$

and the operator is compact in the space $L_2(S)$, so that the equation (7) is Fredholm's one.

The adjoint operator T^* appears in the integral equation of the direct method for the solution of the problem (B). This equation can be obtained from Green's identity (see, e.g., [8]) and the jump relationship for the double layer potentials.

In [8] it was proved that unique solvability of the problems (A) and (B) is equivalent to unique solvability of the integral equations or, in other terms, to the property of the homogeneous boundary integral equations on S

$$-\mu + T\mu = 0, \quad -u' + T^*u' = 0, \tag{8}$$

to have only the trivial solution. We shall denote by Ξ the set of values ν for which the problems and equations (8) have non-trivial solutions.

4. Criteria of uniqueness

We shall use the formalism [6] (see also [7, 8]), based on symmetrization of the integral equations (8). Applying the operator $I-T^*$ to the first equation and $I-T$ to the second one in (8) we arrive at

$$-\mu + \mathfrak{T}\mu = 0, \quad \mathfrak{T} = T + T^* - T^*T, \tag{9}$$
$$-u' + \mathfrak{T}'u' = 0, \quad \mathfrak{T}' = T + T^* - TT^*. \tag{10}$$

It is proved in [8] that the equation (9) has the same set of solutions as the first of the equations (8) and the same is true for the second equation (8) and (10).

It is important to note that $\mathfrak{T}$ and $\mathfrak{T}'$ are compact and, unlike T, self-adjoint operators with real eigenvalues $\lambda_i \in \sigma(\mathfrak{T}) = \sigma(\mathfrak{T}')$. It can be observed that $\langle (I-T^*)(I-T)v, v\rangle = \langle (I-T)v, (I-T)v\rangle \geqslant 0$, where $\langle\cdot,\cdot\rangle$ means scalar product in $L_2(S)$:

$$\langle v, w\rangle = \int_S v\,w\,ds.$$

Thus, $\langle \mathfrak{T}v, v\rangle \leqslant \langle v, v\rangle$ and all eigenvalues are submitted to the inequality $\lambda_i \leqslant 1$. Further we shall use the notation $\lambda_1 = \max\{\lambda_i\}$. It follows from the above that (8) have only the trivial solution if and only if $\lambda_1 < 1$, and non-trivial solutions exist only when $\lambda_1 = 1$. Then, $\Xi = \{\nu : \lambda_1 = 1\}$.

However, this criterion is not suitable for finding examples of non-uniqueness. For this purpose we can use another new criterion of uniqueness suggested in [8]. It is proved (see [7, 8]) that the dimensions of the eigenspaces of $\mathfrak{T}$ and $\mathfrak{T}'$ corresponding to one eigenvalue are equal. Let N_1 be the dimension of the eigenspace corresponding to the maximum eigenvalue λ_1. We introduce $\mu_1^{(i)}$ and ${u'_1}^{(i)}$, $i = 1,\ldots,N_1$, which are eigenfunctions of $\mathfrak{T}$ and $\mathfrak{T}'$, respectively. The following assertion is proved in [8].

The equality $\lambda_1 = 1$ *holds if and only if*

$$\big\langle (I-T^*){u'_1}^{(i)}, \mu_1^{(j)}\big\rangle = 0 \quad \textit{for some } i \textit{ and all } j = 1,\ldots,N_1. \tag{11}$$

Analogously, $\lambda_1 = 1$ if and only if

$$\big\langle (I-T)\mu_1^{(i)}, {u'_1}^{(j)}\big\rangle = 0 \quad \text{for some } i \text{ and all } j = 1,\ldots,N_1. \tag{12}$$

Some disadvantage of the functionals in (11), (12) is the fact that they are defined up to a sign even when we fix $\|\mu_1^{(i)}\| = 1$ and $\|{u'_1}^{(i)}\| = 1$. However we can derive a sufficient condition of non-uniqueness, more convenient for numerical computations.

Suppose that λ_1 is simple and write

$$u'_1 = b\mu_1 + \zeta, \tag{13}$$

where b is some coefficient, ζ is a function belonging to the subspace $L_2(S)$, consisting of functions orthogonal to μ_1.

Let $\lambda_1 \neq 1$. From definition of eigenvalues we have $(I-T)(I-T^*)u_1' = (1-\lambda_1)u_1'$. Applying the operator $I-T^*$ we find $(I-T^*)(I-T)(I-T^*)u_1' = (1-\lambda_1)(I-T^*)u_1'$, or $(I-\mathfrak{T})\hat{\mu}_1 = (1-\lambda_1)\hat{\mu}_1$, where $\hat{\mu}_1 := (I-T^*)u_1'$. Since λ_1 is simple, $\hat{\mu}_1$ differs from μ_1 by a constant non-zero factor. So,

$$(I-T^*)u_1' = c\mu_1,$$

where c is some coefficient. The latter equality holds for $\lambda_1 = 1$ with $c = 0$.

Therefore,

$$\begin{aligned}\big\langle (I-T^*)u_1', u_1' \big\rangle &= \big\langle (I-T^*)u_1', b\mu_1 + \zeta \big\rangle \\ &= b\big\langle (I-T^*)u_1', \mu_1 \big\rangle.\end{aligned}$$

From (13) it follows that $\langle u_1', \mu_1 \rangle = b\langle \mu_1, \mu_1 \rangle$ and,

$$\langle \mu_1, \mu_1 \rangle \big\langle (I-T^*)u_1', u_1' \big\rangle = \langle u_1', \mu_1 \rangle \big\langle (I-T^*)u_1', \mu_1 \big\rangle.$$

Thus we can conclude that if the eigenvalue λ_1 is simple and the conditions

$$\big\langle (I-T^*)u_1', u_1' \big\rangle = 0, \quad \big|\langle \mu_1, u_1' \rangle\big| > 0 \qquad (14)$$

hold, then $\lambda_1 = 1$. An advantage of the functional $\big\langle (I-T^*)u_1', u_1' \big\rangle$ is the fact that it is defined uniquely provided $\|u_1'\| = 1$. The condition (14) allows us to give convincing numerical evidence that for a given geometry non-uniqueness occurs at some ν.

5. Numerical results

In the numerical investigation we use piecewise-constant and cubic spline collocation schemes for approximation of the integral operator T. We shall consider two geometries. First of them (geometry (I) below) consists of two equal ellipses with horizontal and vertical semi-axes a and b, respectively, with centres at depth d, and distance between centres $2l$. Besides, we do numerical investigations for the contour (geometry (II)) defined by the parametric curve $x(t) = a\sin t$, $y(t) = -d + b\cos t - c\cos 2t$, where $t \in [0, 2\pi]$ (see fig. 3).

For the geometries in question we are able to find values ν_k such that the functional $\big\langle (I-T^*)u_1', u_1' \big\rangle$ changes its sign when ν varies through the points. This situation of crossing is numerically stable and confirms finding of parameters corresponding to the non-uniqueness. As examples of the non-uniqueness

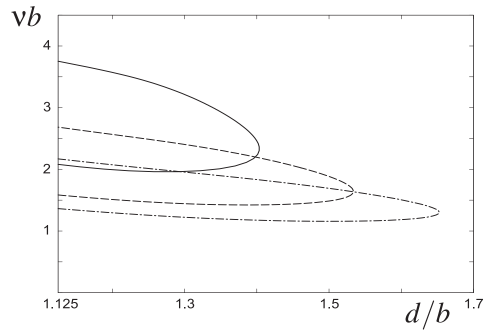

Figure 2: Parameters of non-uniqueness examples for geometry (II), $c/b = 1$ and $a/b = 1$ (solid line), $a/b = 1.5$ (dashed line), $a/b = 2$ (dash-and-point line). The body is totally submerged for $d/b > 1.125$.

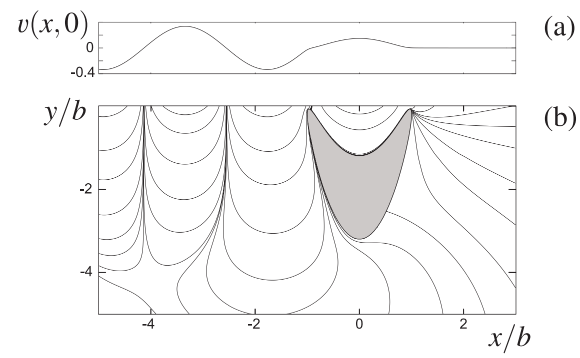

Figure 3: Computations for geometry (II) with $a/b = c/b = 1$, $d/b = 1.2$, and $\nu b = 1.993$: a) values of streamfunction on the free surface $v(x,0)$; b) streamlines $v = \text{const}$.

parameters for the geometry (I) we list the dimensionless values $\nu b = 9.980$ and $\nu b = 16.60$ for $a/b = 0.1$, $d/b = 1.01$, $l/b = 1.2$; $\nu b = 4.059$ and $\nu b = 4.463$ for $a/b = 1$, $d/b = 1.01$, $l/b = 1.1$; $\nu b = 4.006$ and $\nu b = 4.081$ for $a/b = 1$, $d/b = 1.04$, $l/b = 1.1$. Sets of numerically found non-uniqueness parameters for the geometry (II) are presented in fig. 2.

By using the eigenfunction μ_1 it is not difficult to obtain the corresponding solution to the homogeneous problem (1)–(4) in the form $u = V\mu_1$ (where V is the operator defined by (6)). Analogously we can find a streamfunction $v(z) = \int_S \mu_1(\zeta)\, H(z,\zeta)\, ds_\zeta$, where $v(z)$ and $H(z,\zeta)$ are harmonic conjugates in z to $u(z)$ and $G(z,\zeta)$, respectively. Shown in fig. 3b is a picture of the streamlines $v = \text{const}$ for geometry (II) (the contour of the body is one of the streamlines).

It is important to note that in contrast with the water-wave problem (see [5]) for the obtained solutions of homogeneous problem (1)–(4) the Dirichlet energy integral is generally infinite (due to the pres-

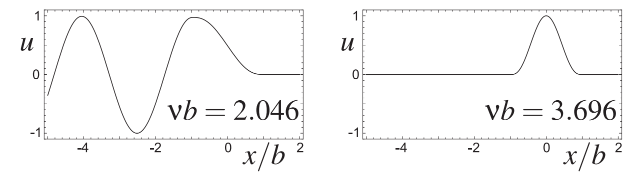

Figure 4: Computations of $u(x,0)$ for geometry (II) with $a/b = c/b = 1$, $d/b = 1.15$.

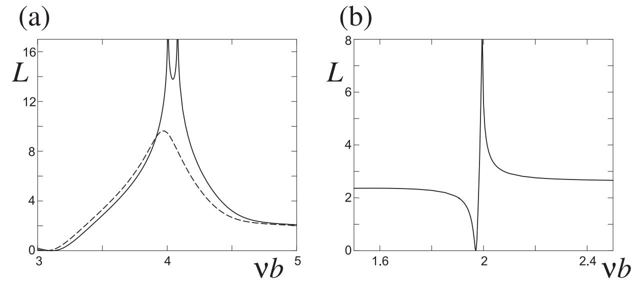

Figure 5: Plots of $L = \log\left(1 - R/(\rho b^2 g)\right)$ against νb: (a) geometry (I), $a/b = 1$, $l/b = 1.1$, and $d/b = 1.04$ (solid line), $d/b = 1.05$ (dashed line); (b) geometry (II), $a/b = c/b = 1$, $d/b = 1.2$.

ence of waves at infinity downstream) and the spectrum of the problem is stable, i.e. the values $\nu_k \in \Xi$ change continuously when a parameter (in fig. 2 it is d/b) is varied.

For the parameters belonging to the curves in fig. 2 it was observed that the downstream waves of corresponding solutions disappear when approaching the upper end-points of the curves. At this limit, when the contour touches the free surface, the solution becomes localized inside the inner 'basin' separated by the contour. In fig. 4 values of $u(x,0)$ for $d/b = 1.15$ (close to the limit value 1.125) are plotted against x/b.

For $\nu \neq \nu_k$ we can solve numerically the non-homogeneous boundary integral equation (7) for the Neumann data $f = U\boldsymbol{n}_x$ and compute the wave resistance

$$R = -\rho \int_S p\,\boldsymbol{n}_x \,\mathrm{d}s,$$

where p is the pressure on the contour, ρ is the density of fluid. Details of computations can be found in [1, § 7.3]; we only mention that it is convenient to express R in terms of coefficients in the far-field asymptotics of the solution to (1)–(4).

In fig. 5 we present in a semilogarithmic scale the wave resistance computed for the geometries (I) and (II) over intervals of νb including the non-uniqueness parameters $\nu b = 4.006$, $\nu b = 4.081$, and $\nu b = 1.993$. For comparison we also show a plot (dashed line in fig. 5a) for the geometry not having non-uniqueness examples.

Our computations give evidence to the fact that the resistance has singularity for $\nu \in \Xi$. Another feature observed in the computation, which may need more interpretation and investigation, is the existence of parameters such that the corresponding wave resistance is very close to zero. The minimum value of L on the curves shown by solid lines in fig. 5 is comparable with the accuracy of computations.

6. Discussion

Non-uniqueness for the two-dimensional Neumann–Kelvin problem describing motion of totally submerged bodies is discovered numerically. The non-uniqueness takes place for isolated, depending on the geometry, values of forward velocity. In order to construct the non-uniqueness examples an algorithm based on the uniqueness criteria, found recently in [8], is developed. Numerical computations of non-uniqueness parameters, corresponding solutions to the homogeneous problem and wave resistance for parameters close to the non-uniqueness values are given and discussed. It is notable that the criteria of uniqueness [8] are also applicable for the three-dimensional case of the problem which assumes further generalization of the present results.

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