APPLICATION OF GENERALIZED FUNCTIONS IN PLANING THEORY

Michael V. Makasyeyev,
Institute of Hydromechanics of NAS Ukraine, Gelyabova str., 8/4, Kiev, 03057, Ukraine,
e-mail: M_Makasyeyev@ukr.net

1. Introduction

The application of generalized functions (distributions) in a theory can be caused by different reasons. One of them is the use of Fourier transform. For example the functions sine and cosine as elementary solutions of surface wave theory problems have only generalized Fourier transformation in mode of delta functions. On the other hand if we use the delta functions for physical problems formulations for description of point sources, point mass, point or instantaneous forces we should formulate the full mathematical problem for generalized functions. The generalized function theory allows to obtain the fundamental solutions of differential equations and to reduce the boundary problem to integral equations. In addition to known advantages related to that all functions become differentiable and all of them have the Fourier transformation, the generalized functions theory gives a new possibility for analytical and qualitative analysis of basis problems.

2. Boundary problem of planing theory

The boundary two-dimensional problem of unsteady planing theory for the velocity potential φ in assumptions of surface waves linearized theory can be written in such form [1−4]:

$$\Delta\varphi(x,y,t)=0,\ y<0, \qquad (2.1)$$

$$N\varphi(x,-0,t)=-p(x,t)-g\eta(x,t), -\infty<x<\infty, \qquad (2.2)$$

$$\varphi_y(x,-0,t)=N\eta(x,t), \quad -\infty<x<\infty, \qquad (2.3)$$

$$\varphi_x,\varphi_y=0,\ y\to-\infty, \qquad (2.4)$$

where $N=\partial/\partial t-V_0\partial/\partial x$, V_0 − constant speed of motion, $p(x,t)=[p_-(x,-0,t)-p_0]/\rho$ − relative pressure on the fluid boundary, $p_-(x,y,t)$ − absolute pressure in the fluid, p_0 − known constant pressure on the free surface, $p(x,t)=0$ at $x<0, x>l(t)$, $l(t)$ − wetted length, $\eta(x,t)$ − the form of free surface at $x\le 0, x>l(t)$ and streamlined surface at $0\le x\le l(t)$. The conditions (2.2) and (2.3) can be not satisfied in usual sense in the points $x=0$ and $x=l(t)$. The illustration of problem definition is shown on fig.1.

At $0\le x\le l(t)$ it will be also used designation $\beta(x,t)=\eta(x,t)$. It is assumed, that $\beta(x,t)=h(t)+x\tan\alpha(t)$, where $h(t)=\beta(0,t)$ − draft of trailing edge, $\alpha(t)$ − trim angle. It is given the abscissa of the centre of mass b. The unknown functions in the problem are $p(x,t)$, $\eta(x,t)$, $l(t)$, $h(t)$, $\alpha(t)$.

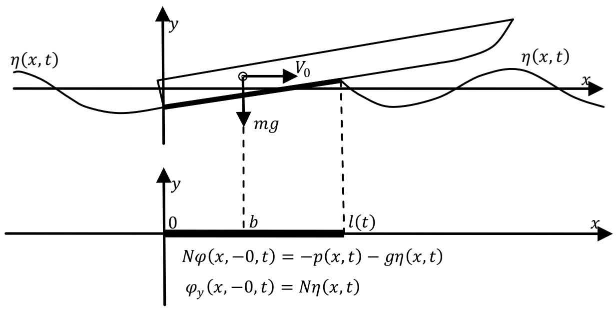

Figure 1. Illustration of problem formulation

3. Initial conditions and radiation condition

Initial conditions are:

$$\varphi(x,y,0)=\varphi_0(x,y),\ \varphi_t(x,y,0)=\varphi_1(x,t), \qquad (3.1)$$

$$\eta(x,0)=\eta_0(x),\ \eta_t(x,0)=\eta_1(x), \qquad (3.2)$$

φ_0, φ_1, η_0, η_1 − known functions.

The radiation condition in problems of the theory of wave motions is set when a solution is suppose as specified – as a rule steady and fully stationary, or steady harmonious. There are no initial conditions in such case. So assigned boundary problem has particular solutions in form free waves with arbitrary amplitudes and moving in all directions. To give a physical sense to the final solution, it is necessary to save in the solution only such waves which move in the directions from a solid body as radiation source.

In the above formulated problem with given initial conditions (3.1) − (3.2) the radiation condition should be fulfil automatically.

4. Solution method

Fourier method for construction of fundamental solutions [5] is used. The velocity potential redefines by zero in the top half-plane − $\varphi(x,y,t)=0$, $y>0$, and generalized function $\varphi=(\varphi,\psi)$ is entered over space of basic functions ψ of slow growth. Designations in brackets for the generalized functions with the view of record simplification are further neglected.

From boundary problem (2.1)−(2.5) by partial integration the following form of Laplacian for generalized function φ is received:

$$\Delta\varphi=-\varphi(x,-0,t)\delta'(y)-\varphi_y(x,-0,t)\delta(y), \qquad (4.1)$$

where $\delta(y)$ and $\delta'(y)$ – delta function and its derivative.

Fourier transform on variable x for the basic functions is defined by the formula $F[\psi(x,)] = \int_{-\infty}^{\infty} \psi(x,)e^{i\lambda x}dx$. Generalized Fourier transformation of function φ denoted as $\Phi(\lambda, y, t) = F[\varphi(x, y, t)]$. In Fourier images the solution of the equation (4.1), taking into account boundary conditions of a problem, is function

$$\Phi(\lambda, y, t) = \overline{N} H(\lambda, t) e^{-|\lambda y|} / |\lambda|, \qquad (4.2)$$

where $\overline{N} = \partial / \partial t + i\lambda V_0$, $H(\lambda, t) = F[\eta(x, t)]$.

On basis of (2.2), (2.3) it is possible to receive a functional relation

$$\left(\overline{N}^2 / |\lambda| + g\right) H(\lambda, t) = -P(\lambda, t), \qquad (4.3)$$

where $\overline{N}^2 = \partial^2 / \partial t^2 + 2i\lambda V_0 \, \partial / \partial t - \lambda^2 V_0^2$ and $P(\lambda, t) = F[p(x, t)]$.

The function

$$H(\lambda, t) = |\lambda| \int_0^t P(\lambda, \tau) e^{i\lambda V_0(\tau - t)} D(\lambda, \tau - t) d\tau + G_0(\lambda, t) e^{-i\lambda V_0 t} \qquad (4.4)$$

satisfies relation (4.3) and initial conditions (3.2). There are

$$D(\lambda, t) = \left(\sin \sqrt{g|\lambda|} t\right) / \sqrt{g|\lambda|},$$

$$G_0(\lambda, t) = H_0(\lambda) \frac{\partial D}{\partial t}(\lambda, t) + H_1(\lambda) D(\lambda, t),$$

$$H_0(\lambda) = F[\eta_0(x)](\lambda), \; H_1(\lambda) = F[\eta_1(x)](\lambda).$$

The inverse Fourier transformation of (4.4) gives in space of originals the formula for liquid boundary form calculating by means of pressure distribution:

$$\eta(x, t) = \int_0^{l(t)} \int_0^t p(s, \tau) K_0(x - s - V_0(\tau - t), \tau - t) d\tau ds + \zeta_0(x + V_0 t, t), \; -\infty < x < \infty, \qquad (4.5)$$

where [6]

$$K_0(x, t) = F_x^{-1}\left[\sqrt{\frac{|\lambda|}{g}} \sin \sqrt{g|\lambda|} t\right] = -\frac{t}{2\pi x^2} - \sqrt{\frac{g}{2\pi}} \frac{1}{|x|^{3/2}} [\operatorname{sgn} x \, S(\mu) f_{01}(\mu) - C(\mu) f_{02}(\mu)], \qquad (4.6)$$

$S(x) = \int_0^x \sin \frac{\pi}{2} s^2 ds$ and $C(x) = \int_0^x \cos \frac{\pi}{2} s^2 ds$ – sine - and cosine Fresnel's integrals,

$$f_{01}(\mu) = \operatorname{sgn} x \sin \frac{\pi}{2} \mu^2 + \frac{\mu^2}{\pi} \cos \frac{\pi}{2} \mu^2,$$

$$f_{02}(\mu) = \cos \frac{\pi}{2} \mu^2 - \frac{\mu^2}{\pi} \operatorname{sgn} x \sin \frac{\pi}{2} \mu^2,$$

$$\mu = \sqrt{\frac{g}{2\pi}} \frac{t}{\sqrt{|x|}}, \; \zeta_0(x, t) = F_x^{-1}[G_0(\lambda, t)].$$

Relation (4.5) can be used to calculate of free surface form, and to define of pressure function, when it is satisfying on the interval $(0, l(t))$, where the function $\eta(x, t) = \beta(x, t)$ is partially known. In the most cases is more efficient to use the equation corresponding to a boundary condition (2.4) for determination of pressure function:

$$\int_0^{l(t)} \int_0^t p(s, \tau) K_1(x - s - V_0(\tau - t), \tau - t) d\tau ds + \zeta_1(x + V_0 t, t) = N\beta(x, t), \qquad (4.7)$$

where [6]

$$K_1(x, t) = F^{-1}\left[|\lambda| \cos \sqrt{g|\lambda|} t\right] = -\frac{1}{\pi x^2} - \sqrt{\frac{g}{2\pi}} \frac{t}{|x|^{5/2}} [S(\mu) f_{11}(\mu) + \operatorname{sgn} x \, C(\mu) f_{12}(\mu)], \qquad (4.8)$$

$$f_{11}(\mu) = \frac{3}{2} \cos \frac{\pi}{2} \mu^2 - \frac{\mu^2}{\pi} \operatorname{sgn} x \sin \frac{\pi}{2} \mu^2,$$

$$f_{12}(\mu) = \left(\frac{3}{2} \sin \frac{\pi}{2} \mu^2 + \frac{\mu^2}{\pi} \cos \frac{\pi}{2} \mu^2\right) \operatorname{sgn} x,$$

$$\zeta_1(x, t) = F_x^{-1}\left[\frac{\partial G_0}{\partial t}(\lambda, t)\right](x, t).$$

To define wetted length $l(t)$ and trim angle $\alpha(t)$ it is necessary to complement the integral equation (4.7) with additional equations of solid body dynamics as it was made in the case of steady motion [7].

When $\lambda = 0$ and $G_0(\lambda, t) = 0$, we have $H(0, t) = 0$ from (4.4) on the strength $P(0, t) = \int_{-\infty}^{\infty} p(x, t) dx < \infty$ by any t. The equality $H(0, t) = 0$ conforms to $\int_{-\infty}^{\infty} \eta(x, t) dx = 0$ in the space of Fourier originals and this implies $\int_{-\infty}^{\infty} \varphi_y(x, -0, t) dx = 0$. It is the condition of one-valued solvability of Neumann problem.

5. Relations for initial conditions

Let's write the relations for three main types of initial conditions for unsteady problems of planing theory – undisturbed surface, wave surface and steady motion on undisturbed surface.

5.1. Undisturbed surface

It is supposed, that before the beginning of motion the liquid is quiet:

$$\eta_0(x) = 0, \; \eta_1(x) = 0, \; p_0(x) = 0,$$

then, respectively,

$$H_0(\lambda) = 0, \; H_1(\lambda) = 0, \; P_0(\lambda) = 0. \qquad (5.1)$$

5.2. Wave surface

It is supposed that in the initial time on liquid surface there is no perturbation except the regular steady-state waves:

$$\eta(x,t) = \operatorname{Re}\eta(x)e^{ikt}, \qquad (5.2)$$

where $\eta(x)$ − complex amplitude function, k − oscillation frequency. In this case k is oscillation frequency in coordinate system which moves with velocity V_0 in a positive direction of axis x. It is apparent from the point of view of the moving observer − the same waves will have different seeming frequency at different speed of motion. The real frequency is reasonably to set in the absolute fixed coordinate system. The necessary relations can be received from the initial problem (2.1)–(2.4), if suppose $V_0 = 0$ and $p(x,t) = 0$ and after solution to return to the moving coordinates with corresponding replacement of the variable.

Let k_0 and $\eta_0(x,t)$ − oscillation frequency and form of a free surface in absolute fixed coordinate system. Then

$$\eta_0(x,t) = \operatorname{Re}\eta_0(x)e^{ik_0t}. \qquad (5.3)$$

For determinacy it is supposed $k_0 \geq 0$. Relation (4.3) for the Fourier transform $H_0(x)$ of function $\eta_0(x)$ at assumptions $V_0 = 0$ and $p(x,t) = 0$ will take on form (here and further the sign Re for simplification is not written, in final results it is necessary to take the real part)

$$\left(-\frac{k_0^{\,2}}{|\lambda|} + g\right)H_0(\lambda) = 0.$$

So,

$$H_0(\lambda) = A_0\delta(\lambda - \omega_0) + B_0\delta(\lambda + \omega_0), \qquad (5.4)$$

where A_0, B_0 − complex constants, $\omega_0 = k_0^{\,2}/g$. Fourier inversion of (5.4) gives

$$\eta_0(x) = \overline{A}_0 e^{-i\omega_0 x} + \overline{B}_0 e^{i\omega_0 x},$$

where $\overline{A}_0 = A_0/2\pi$, $\overline{B}_0 = B_0/2\pi$. Then

$$\eta_0(x,t) = \operatorname{Re}\left[\overline{A}_0 e^{-i(\omega_0 x - k_0 t)} + \overline{B}_0 e^{i(\omega_0 x + k_0 t)}\right].$$

It is the superposition of two waves, moving in different sides. The wave with amplitude $\overline{A}_0$ moves in a positive direction of axis x (to the right) with velocity $V_A = k_0/\omega_0 = g/k_0$, wave with amplitude $\overline{B}_0$ − to the left with velocity $V_B = -g/k_0$.

Conversion to the coordinate system which moves with velocity V_0 is come to be out by change of variables $x = \overline{x} + V_0 t$:

$$\eta_0(\overline{x} + V_0 t, t) =$$
$$= \operatorname{Re}\left\{\overline{A}_0 e^{-i[\omega_0\overline{x} + (\omega_0 V_0 - k_0)t]} + \overline{B}_0 e^{i[\omega_0\overline{x} + (\omega_0 V_0 + k_0)t]}\right\}. \qquad (5.5)$$

In the moving coordinate system the relative velocity of wave with amplitude $\overline{A}_0$ now is $\overline{V}_A = -V_0 + g/k_0$, the velocity of wave with amplitude $\overline{B}_0$ thus $\overline{V}_B = -V_0 - g/k_0$. If $V_0 < g/k_0$, then $\overline{V}_A > 0$, that is the wave with amplitude $\overline{A}_0$ moves towards movement of observation point, overtaking it. At $V_0 = g/k_0$ the relative speed of a wave with amplitude $\overline{A}_0$ is equal to zero that is the reference system moves with the velocity of this wave. If $V_0 > g/k_0$, then $\overline{V}_A < 0$, that is the wave with amplitude $\overline{A}_0$ moves towards to the body. As always $\overline{V}_B < 0$, the wave with amplitude $\overline{B}_0$ is always oncoming.

By selection of constants $\overline{A}_0$ and $\overline{B}_0$ various modes of initial waves are set, including standing.

Comparison (5.5) and (5.3) shows, that the representation of waves parameters in fixed coordinate system in the form of (5.3) gives in moving coordinates expression which differs from (5.3) − it is the composition from two waves with different frequencies $k_1 = k_0 - \omega_0 V_0$ and $k_2 = k_0 + \omega_0 V_0$. Frequency k_2 is positive, k_1 can be positive and negative. Wave with frequency k_2 is oncoming, with frequency k_1 − following. At $k_1 < 0$ the wave moves faster than the body, at $k_1 = 0$ wave and body velocities coincide.

It is obvious, that parameters of initial waves in moving coordinates can be set also in the form (5.2), and then to receive formulas for the expression of wave parameters in fixed coordinate system. Thus for determinacy it is necessary to agree with, whether there can be negative frequency k. It is necessary to note, that even in fixed coordinate system by frequency sign it is possible to identify the direction of wave propagation.

5.3 Steady motion on undisturbed surface

In this case assumed that initial motion mode is steady-state, i.e. all magnitudes are time-independent.

Functions $\eta_0(x)$ and $p_0(x)$ can be determined by two ways. The first − from initial problem provided that the motion mode is already stationary and there is no dependencies from time, and the second − from initial unsteady problem with zero initial conditions at great values of time. The first is connected with necessity of radiation conditions performance, the second is based on the use of the equation (3.3) and on the limiting process at $t \to \infty$. Let's show, that both ways give the same result.

Accordingly to the first way, $\eta(x,t) = \eta_0(x)$, $p(x,t) = p_0(x)$. Relation (3.3) for Fourier transforms of this function will be written as

$$\left(|\lambda|V_0^{\,2} - g\right)H_0(\lambda) = P_0(\lambda). \qquad (5.6)$$

This relation satisfies the function which is the product of $P_0(\lambda)$ on the solution in generalized functions of algebraic equation with left part (5.6) and right in the form of unit function:

$$H_0(\lambda)=\frac{P_0(\lambda)}{V_0^2}\left(reg\frac{1}{|\lambda|-\nu}\right)+ \\ +\left[AP_0(\nu)+A_0\right]\delta(\lambda-\nu)+ \\ +\left[BP_0(-\nu)+B_0\right]\delta(\lambda+\nu), \qquad (5.7)$$

where reg means regularization of function, A, B, A_0, B_0 − complex constants, which specify amplitudes of free waves on space of originals, $\nu = g/V_0^2$. Constants A and B are defined from radiation condition, which gives $A=\pi i$, $B=-\pi i$, A_0 and B_0 − amplitudes of independent waves, if they are set. Further assumed $A_0=0$, $B_0=0$.

In result (5.7) rewritten as

$$H_0(\lambda)=\frac{P_0(\lambda)}{V_0^2}\left(reg\frac{1}{|\lambda|-\nu}\right)+ \\ +\frac{\pi i}{V_0^2}\left[P_0(\nu)\delta(\lambda-\nu)-P_0(-\nu)\delta(\lambda+\nu)\right]. \qquad (5.8)$$

Accordingly to the second way it can be supposed, that in the initial time on the undisturbed liquid surface appeared instantly a perturbation of pressure corresponding, for example, to hydrostatic at $V_0=0$. The subsequent evolution of liquid boundary and pressure distribution is defined by means of the equations, corresponding (3.5) and (3.9). Solution can be constructed sequentially on time layers.

As far as during all motion time there are no any external perturbations, it is necessary to expect, that at constant load with large values of time, the acceleration of mass centre of the planing body will be equal to zero. It is possible to show, that from this assumption follows, that $P(\lambda,t)=P(\lambda)$. Then the relation (4.4) can be written as

$$H(\lambda,t)=\sqrt{\frac{|\lambda|}{g}}P(\lambda)\int_0^t e^{i\lambda V_0(\tau-t)}\sin\sqrt{g|\lambda|}(\tau-t)d\tau .$$

Representing the sine of exponential function, it can be transformed to a form

$$H(\lambda,t)= \\ =-\frac{i}{2\sqrt{g}}P(\lambda)\int_{-\infty}^{\sqrt{|\lambda|}t}\theta(\tau)\left[e^{-i\tau\left(V_0\operatorname{sgn}\lambda\sqrt{|\lambda|}-\sqrt{g}\right)}- \right. \\ \left. -e^{-i\tau\left(V_0\operatorname{sgn}\lambda\sqrt{|\lambda|}+\sqrt{g}\right)}\right]d\tau , \qquad (5.9)$$

where $\theta(\tau)$ − Heaviside's function. In a limit at $t\to\infty$ and $\lambda\neq 0$ the integral in (5.9) will be the Fourier transform of sum of the Heaviside's functions. As it known [5],

$$\int_{-\infty}^{\infty}\theta(\tau)e^{i\tau\xi}d\tau = ireg\left(\frac{1}{\zeta}\right)+\pi\delta(\zeta).$$

Then on basis of (5.9) we obtain

$$\lim_{t\to\infty}H(\lambda,t)=reg\frac{P(\lambda)}{|\lambda|V_0^2-g}+ \\ +\frac{\pi i}{2\sqrt{g}}P(\lambda)\left[\delta\left(V_0\operatorname{sgn}\lambda\sqrt{|\lambda|}-\sqrt{g}\right)- \right. \\ \left. -\delta\left(V_0\operatorname{sgn}\lambda\sqrt{|\lambda|}+\sqrt{g}\right)\right],$$

where delta-functions are focussed in zeroes of functions which are write in brackets as arguments. Each of these functions is continuous with all of it derivatives and have one simple root. Therefore we can use the decomposition formulas of delta-functions and write

$$\delta\left(V_0\operatorname{sgn}\lambda\sqrt{|\lambda|}\mp\sqrt{g}\right)=\frac{2\sqrt{g}}{V_0^2}\delta(\lambda\mp\nu).$$

Consequently

$$\lim_{t\to\infty}H(\lambda,t)=\frac{P(\lambda)}{V_0^2}\left(reg\frac{1}{|\lambda|-\nu}\right)+ \\ +\frac{\pi i}{V_0^2}\left[P(\nu)\delta(\lambda-\nu)-P(-\nu)\delta(\lambda+\nu)\right]. \qquad (5.10)$$

We can see that right-hand members of (5.10) and (5.8) are identical.

6. Planing on regular waves

In this case the initial conditions will be (5.5). The formula for the free surface at $t\to\infty$ can be obtain from (4.4) in the same technique that was obtained (5.10) for steady motion.

References

1. Bessho M., Komatsu M., 1984, Two-Dimensional Unsteady Planing Surface // Journal of Ship Research, 28, № 1, pp. 18-28c.
2. Ulstein T., Faltinsen O. M., 1996, Two-Dimensional Unsteady Planing // Journal of Ship Research, 40, № 3, pp. 200-210.
3. Kashiwagi M., 2003, A Flow Model for a Displacement-Type Fast Ship with Shallow Draft in Regular Waves // Twenty-Fourth Symposium on Naval Hydrodynamics, Fukuoka, Japan, 11 p.
4. Sedov L.I. Flat problems of hydrodynamics and aerodynamics. – Moscow, Nauka. 1980. – 448 p. (In Russian)
5. Vladimirov V.S. Equations of Mathematical Physics. − Moscow, Nauka. 1988. – 512 p. (In Russian)
6. Brychkov Y.A., Prudnikov A.P. Integral transformations of generalized functions. – Moscow, Nauka. 1977. – 287 p. (In Russian)
7. Makasyeyev M.V. Steady motion of plate on the surface of heavy fluid at given load and free trim angle // Applied hydromechanics. 2003. Vol. 5(77). № 2. P. 73–75. (In Russian)