Analytical solution for the capillary–gravity waves due to an oscillating Stokeslet

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I. INTRODUCTION

Generation of the Cauchy–Poisson waves (CPW) in a stationary fluid and the Neumann–Kelvin waves (NKW) in a uniformly running stream by the initial elevation and impulse at the surface of a fluid was usually considered in the framework of linear potential theory. The singular behavior, as predicted by the potential theory, of infinite amplitudes for the CPW in the near region, and for the diverging component of the NKW near the moving path of the pressure point, can be removed by the inclusion of the viscosity [1–3] or the surface tension [4, 5].

Preliminary studies on the combined effects of viscosity and surface tension on the free-surface waves in Stokes and Oseen flows have been performed by Chen *et al.* [6] and Chen & Lu [7], respectively. Asymptotic solutions for the interfacial capillary–gravity waves due to an instantaneous fundamental singularity in a system of two semi-infinite fluids were recently provided by Lu & Ng [8]. In this paper, we derive the analytical solution with the aid of asymptotic analysis for the capillary–gravity waves due to an oscillating point force in a Stokes flow.

II. MATHEMATICAL FORMULATION

The governing equations are the continuity equation

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

and the singularly forced Stokes equations

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{F} \exp(\mathrm{i}\Omega t)\delta(\mathbf{x} - \mathbf{x}_0), \tag{2}$$

where $\mathbf{u} = (u, v, w)$ is the disturbed velocity field, P is the hydrodynamic pressure, ρ and μ are the density and viscosity of the fluid, respectively. $\mathbf{F} \exp(\mathrm{i}\Omega t)\delta(\mathbf{x} - \mathbf{x}_0)$ is the singular force located at position $\mathbf{x}_0$, where $\mathbf{F} = (0, 0, F)$, Ω the oscillating frequency, $\delta(\ )$ the Dirac delta function, $\mathbf{x} = (x, y, z)$ the field point, and $\mathbf{x}_0 = (0, 0, -h_0)$ the source point with $h_0 > 0$.

The linearized boundary conditions on the undisturbed free surface ($z = 0$) are

$$\frac{\partial \eta}{\partial t} = w, \tag{3}$$

$$\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0, \tag{4}$$

$$\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0, \tag{5}$$

$$p - 2\mu \frac{\partial w}{\partial z} + T \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) = 0, \tag{6}$$

where $p = P - \rho g \eta$ is the total pressure, η the elevation of the free surface and T the coefficient of the surface tension. Equation (3) states that no fluid particles cross the free surface. Equation (4) represents the vanishing of shearing stress in the x direction while Eq. (5) in the y direction. Equation (6) denotes the continuity of normal stresses on the free surface. In addition, the initial values of the velocity, the hydrodynamic pressure and the free-surface elevation are taken to be those of the quiescent fluid, that is,

$$\mathbf{u}|_{t=0} = \mathbf{0}, \quad P|_{t=0} = 0, \quad \eta|_{t=0} = 0. \tag{7}$$

Governing equations (1) and (2) together with conditions (3) to (7) form a general initial–boundary-value problem associated with a fundamental singularity. The mathematical procedure for dealing with Eqs. (1) to (7) in the remainder of this section is similar to that in Ref. [3] and is repeated here for the sake of completeness. Next we regard the disturbed flow $(\mathbf{u}, P)$ as the sum of an unbounded singular Stokes flow $(\mathbf{u}_\mathrm{s}, P_\mathrm{s})$ which represents the effect of the singular force and a bounded regular Stokes flow $(\mathbf{u}_\mathrm{r}, P_\mathrm{r})$ which represents the influence of the free surface. Thus, we write

$$\{\mathbf{u}, P\} = \{\mathbf{u}_\mathrm{s}(\mathbf{x}, t; \mathbf{x}_0), P_\mathrm{s}(\mathbf{x}, t; \mathbf{x}_0)\} + \{\mathbf{u}_\mathrm{r}(\mathbf{x}, t), P_\mathrm{r}(\mathbf{x}, t)\}. \tag{8}$$

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As is well known, any continuous vector can be taken as the sum of an irrotational and a solenoidal vector, $\mathbf{u}_{\mathrm{r}} = \nabla\Phi + \mathbf{V}_{\mathrm{t}}$, where Φ, a scalar potential function, represents an irrotational flow while $\mathbf{V}_{\mathrm{t}}$ represents a rotational flow. Thus, we have

$$\nabla^2\Phi = 0, \tag{9}$$

$$P_{\mathrm{r}} = -\rho\frac{\partial\Phi}{\partial t} + f(t), \tag{10}$$

$$\nabla\cdot\mathbf{V}_{\mathrm{t}} = 0, \tag{11}$$

$$\frac{\partial\mathbf{V}_{\mathrm{t}}}{\partial t} = \nu\nabla^2\mathbf{V}_{\mathrm{t}}, \tag{12}$$

where $\nu = \mu/\rho$, $f(t)$ is an undetermined function of t. Therefore, the boundary conditions can be expressed in terms of $\mathbf{u}_{\mathrm{s}}$, P_{s}, Φ and $\mathbf{V}_{\mathrm{t}}$ on the undisturbed free surface $(z = 0)$,

$$\frac{\partial\eta}{\partial t} - \left(\frac{\partial\Phi}{\partial z} + w_{\mathrm{t}}\right) = w_{\mathrm{s}}, \tag{13}$$

$$2\frac{\partial^2\Phi}{\partial x\partial z} + \frac{\partial u_{\mathrm{t}}}{\partial z} + \frac{\partial w_{\mathrm{t}}}{\partial x} = -\left(\frac{\partial u_{\mathrm{s}}}{\partial z} + \frac{\partial w_{\mathrm{s}}}{\partial x}\right), \tag{14}$$

$$2\frac{\partial^2\Phi}{\partial y\partial z} + \frac{\partial v_{\mathrm{t}}}{\partial z} + \frac{\partial w_{\mathrm{t}}}{\partial y} = -\left(\frac{\partial v_{\mathrm{s}}}{\partial z} + \frac{\partial w_{\mathrm{s}}}{\partial y}\right), \tag{15}$$

$$\frac{\partial\Phi}{\partial t} + g\eta + 2\nu\left(\frac{\partial^2\Phi}{\partial z^2} + \frac{\partial w_{\mathrm{t}}}{\partial z}\right) - \tau\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) = \frac{P_{\mathrm{s}} + f(t)}{\rho} - 2\nu\frac{\partial w_{\mathrm{s}}}{\partial z}, \tag{16}$$

where $\tau = T/\rho$, $(u_{\mathrm{s}}, v_{\mathrm{s}}, w_{\mathrm{s}})$ and $(u_{\mathrm{t}}, v_{\mathrm{t}}, w_{\mathrm{t}})$ are the components of $\mathbf{u}_{\mathrm{s}}$ and $\mathbf{V}_{\mathrm{t}}$, respectively. Next the function $f(t) = -\rho\Phi(x, y, 0, 0)\delta(t)$ is imposed in order to satisfy the initial conditions (7).

III. FORMAL SOLUTION AND ASYMPTOTIC REPRESENTATION

The fundamental solution of Eqs. (1)–(2), referred to as the oscillating Stokeslet, can be written as [8]

$$\mathbf{u}_{\mathrm{s}} = \frac{\mathbf{F}\cdot(\nabla\nabla - \mathbf{I}\nabla^2)}{16\pi^3\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\iint_{-\infty}^{\infty}\mathrm{d}\alpha\mathrm{d}\beta \times\frac{\exp(f)}{s(s-\mathrm{i}\Omega)}\left[\frac{1}{k}\exp(-k|z+h_0|) -\frac{1}{b}\exp(-b|z+h_0|)\right], \tag{17}$$

$$P_{\mathrm{s}} = -\frac{\mathbf{F}\cdot\nabla}{16\pi^3\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\iint_{-\infty}^{\infty}\mathrm{d}\alpha\mathrm{d}\beta \times\frac{1}{k(s-\mathrm{i}\Omega)}\exp(-k|z+h_0| + f), \tag{18}$$

where $\mathbf{I}$ is a unit tensor of rank two and $f = \mathrm{i}\alpha x + \mathrm{i}\beta y + st$, $k = \sqrt{\alpha^2+\beta^2}$, $b = \sqrt{s/\nu + k^2}$.

By taking a Laplace–Fourier transform, the solution for the surface elevation is given as

$$\eta = \frac{F}{8\pi^3\mathrm{i}\rho}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\iint_{-\infty}^{\infty}\frac{A\exp(f)}{s(s-\mathrm{i}\Omega)D}\mathrm{d}\alpha\mathrm{d}\beta, \tag{19}$$

where

$$A(s,k) = k(s+2\nu k^2)\exp(-kh_0) - 2\nu k^3\exp(-bh_0), \tag{20}$$

$$D(s,k) = \omega^2 + (s+2\nu k^2)^2 - 4\nu^2k^3b, \tag{21}$$

$$\omega(k) = \sqrt{gk + \tau k^3}. \tag{22}$$

With a change of variables $\{x, y\} = \{R\cos\theta, R\sin\theta\}$, $\{\alpha,\beta\} = \{k\cos\phi, k\sin\phi\}$, we may re-write Eq. (19) as

$$\eta = \frac{F}{4\pi^2\rho\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_0^{\infty}\frac{kA\mathrm{J}_0(kR)\exp(st)}{s(s-\mathrm{i}\Omega)D}\mathrm{d}k, \tag{23}$$

where $\mathrm{J}_0(kR)$ is the Bessel function of the first kind of order zero.

The exact evaluation of the integral expression (23) for all instants in general can only be performed numerically. In order to obtain analytically the principal physical features of the wave motion, it is necessary to adopt the asymptotic analysis for the wave integral. Next, the asymptotic behavior of Eq. (23) shall be studied for large t with R/t held fixed.

As the first stage, we may replace $\mathrm{J}_0(kR)$ in Eq. (23) by its asymptotic formula for large kR,

$$\mathrm{J}_0(kR) \sim \left(\frac{2}{\pi kR}\right)^{1/2}\cos\left(kR - \frac{\pi}{4}\right). \tag{24}$$

Then the inversion of the Laplace transform in Eq. (23) is evaluated by use of the Cauchy residue theorem. It is easily seen that the integrand in Eq. (23) has four poles with respect to s, denoted by s_j with $j = 1, 2, 3, 4$,

$$s_j = (-1)^{j+1}\mathrm{i}\omega - 2\nu k^2 + o(\nu k^2), \quad (j = 1, 2), \tag{25}$$

$$s_3 = 0, \quad s_4 = \mathrm{i}\Omega. \tag{26}$$

By taking a contour integration in the complex s plane, Eq. (23) can be represented by

$$\eta = \eta_{\mathrm{S}} + \eta_{\mathrm{T}}, \tag{27}$$

where

$$\eta_{\rm S} = \frac{F}{2\pi\rho}\sum_{n=1}^{2}\int_0^\infty {\rm d}k\left(\frac{k}{2\pi R}\right)^{1/2}\frac{A_{\rm S}}{{\rm i}\Omega D_{\rm S}} \times \exp\left[(-1)^{n+1}{\rm i}\left(kR-\frac{\pi}{4}\right)+{\rm i}\Omega t\right], \quad (28)$$

$$\eta_{\rm T} \sim \frac{F}{4\pi\rho}\sum_{n=1}^{2}\sum_{j=1}^{2}\int_0^\infty {\rm d}k\left(\frac{k}{2\pi R}\right)^{1/2} \times \exp\left(-2\nu k^2 t+{\rm i}t\Theta_{nj}\right) \times \frac{(-1)^{j+1}s_{3-j}(s_{3-j}-{\rm i}\Omega)A_{{\rm T}j}}{{\rm i}\omega s_1 s_2 D_{\rm S}}, \quad (29)$$

$$\{A_{\rm S}(k), D_{\rm S}(k)\} = \{A(s,k), D(s,k)\}|_{s={\rm i}\Omega}, \quad (30)$$

$$A_{{\rm T}j}(k) = A(s,k)|_{s=s_j}, \quad (31)$$

$$\Theta_{nj} = (-1)^{n+1}\frac{1}{t}\left(kR-\frac{\pi}{4}\right)+(-1)^{j+1}\omega. \quad (32)$$

In accordance with the theorem developed by Lighthill [9, p. 52] for the Fourier-type integrals, the major contribution to the integrals in Eq. (28) for a large distance comes from the zeros of $D_{\rm S}$. For small ν, the asymptotic solution of $D_{\rm S}=0$, denoted by k_Ω, is readily given by

$$k_\Omega(\Omega,\tau) = k_0 - \frac{4{\rm i}\nu k_0^2\Omega}{g+2\tau k_0^2}+o(\nu), \quad (33)$$

where

$$k_0(\Omega,\tau) = \frac{1}{\tau}\left(\frac{a}{36}\right)^{1/3} - g\left(\frac{2}{3a}\right)^{1/3}, \quad (34)$$

$$a(\Omega,\tau) = 9\tau^2\Omega^2 + \left(12g^3\tau^3+81\tau^4\Omega^4\right)^{1/2}. \quad (35)$$

A straightforward application of Lighthill's theorem yields the asymptotic solutions for $\eta_{\rm S}$ at large R

$$\eta_{\rm S} \sim -\frac{FA_\Omega}{\rho\Omega}\left(\frac{k_\Omega}{2\pi R}\right)^{1/2} \times \exp\left[-{\rm i}\left(k_\Omega R-\frac{\pi}{4}\right)+{\rm i}\Omega t\right], \quad (36)$$

where

$$A_\Omega(\Omega,\tau) = A_{\rm S}(k)|_{k=k_\Omega}. \quad (37)$$

For the k integration in Eq. (29), the method of stationary phase is used for large t with R/t held fixed. The dominant contribution to the integral in Eq. (29) stems from the stationary points of the oscillatory factors of the integrand. It is easily seen that the stationary points for Θ_{12} and Θ_{21} are the same, which can be determined by

$$\frac{\partial\Theta_{12}}{\partial k} = \frac{R}{t} - C_{\rm g} = \frac{R}{t} - \frac{1}{2}(G+3k^2)\left(\frac{\tau}{Gk+k^3}\right)^{1/2} = 0, \quad (38)$$

where $G=g/\tau$, $C_{\rm g}(k)=\partial\omega/\partial k$ is the group velocity. It is well known that there exists a minimum group velocity $C_{\rm gmin}=C_{\rm g}(k_{\rm c})$, which is given as

$$C_{\rm gmin}(\tau) = (\sqrt{3}-1)\sqrt{\frac{3\tau}{2}}\left(\frac{G}{2\sqrt{3}-3}\right)^{1/4}, \quad (39)$$

$$k_{\rm c}(\tau) = \sqrt{\left(\frac{2}{\sqrt{3}}-1\right)G}. \quad (40)$$

When $R/t > C_{\rm gmin}$, Eq. (38) has two real positive roots, $k_1(R/t,\tau)$ and $k_2(R/t,\tau)$ with $0<k_1<k_2<+\infty$. Approximate solutions for k_1 and k_2 have been provided by Chen and Duan [5, Eqs. (6) & (8)] for the free-surface capillary–gravity waves in an inviscid fluid. As it is, the exact solutions for k_1 and k_2 can readily be obtained as [8, Eq. (39)]

$$k_m(R/t,\tau) = \frac{1}{36}[X+\sqrt{Q_4}+(-1)^m\sqrt{2Q_5}], \quad (m=1,2), \quad (41)$$

where

$$X = 4R^2/(\tau t^2), \quad (42)$$

$$Q_1 = -6912G^2+288GX^2+X^4, \quad (43)$$

$$Q_2 = -576G^3+36(GX)^2+\sqrt{3G^3X^2Q_1}, \quad (44)$$

$$Q_3 = 48G-X^2, \quad (45)$$

$$Q_4 = 12(3Q_2)^{1/3}+12GQ_3(9/Q_2)^{1/3} - 144G+X^2, \quad (46)$$

$$Q_5 = -6(3Q_2)^{1/3}-6GQ_3(9/Q_2)^{1/3} - 144G+X^2+(432GX+X^3)/Q_4^{1/2}. \quad (47)$$

The comparison between the exact solutions in Eq. (41) and the asymptotic solutions of Chen and Duan [5, Eqs. (6) & (8)] was illustrated by Lu and Ng [8, Fig. 2].

When $R/t > C_{gmin}$, a straightforward application of the stationary phase approximation yields the formal expression for the viscous capillary–gravity wave profile for large t with R/t held fixed,

$$\eta_{\rm T} = \frac{F}{2\pi\rho}\sum_{m=1}^{2}\exp(-2\nu k_m^2 t)\left(\frac{k_m^3}{R|\omega_m''|t}\right)^{1/2} \times \frac{\cos\varphi_m+{\rm i}\omega^{-1}\Omega\sin\varphi_m}{\omega_m^2-\Omega^2}+o(\nu/t), \quad (48)$$

where

$$\omega_m = \omega(k_m) = \sqrt{gk_m + \tau k_m^3}, \quad (49)$$

$$\omega_m'' = \frac{\partial^2\omega(k_m)}{\partial k^2} = \frac{1}{4}(-G^2 + 6Gk_m^2 + 3k_m^4) \times \left[\frac{\tau}{(Gk_m + k_m^3)^3}\right]^{1/2}, \quad (50)$$

$$\varphi_m = k_m R - \omega_m t + [(-1)^{m+1} - 1]\pi/4. \quad (51)$$

As R/t decreases to approach C_{gmin}, k_1 and k_2 will go together toward the same limit k_c while ω_m'' tends to zero. Accordingly, Eq. (48) predicts that the wave amplitudes will increase without bounds. In this case, according to the Scorer method of stationary-phase [10], Eq. (29) can be approximated by

$$\eta_T = \frac{F\exp(-2\nu k_c^2 t)}{2\rho}\left(\frac{k_c^3}{2\pi R}\right)^{1/2}\left(\frac{2}{\omega_c''' t}\right)^{1/3}\text{Ai}(b) \times \frac{\cos\varphi_c + \mathrm{i}\omega^{-1}\Omega\sin\varphi_c}{\omega_c^2 - \Omega^2} + o(\nu/t), \quad (52)$$

where

$$\omega_c''' = \frac{\partial^3\omega(k_c)}{\partial k^3} = \frac{3}{8}(G^3 + 5G^2k_c^2 - 5Gk_c^4 - k_c^6) \times \left[\frac{\tau}{(Gk_c + k_c^3)^5}\right]^{1/2}, \quad (53)$$

$$b = (\omega_c' t - R)\left(\frac{2}{\omega_c''' t}\right)^{1/3}, \quad (54)$$

$$\omega_c = \omega(k_c) = \sqrt{gk_c + \tau k_c^3}, \quad (55)$$

$$\varphi_c = k_c R - \omega_c t - \pi/4, \quad (56)$$

and $\text{Ai}(\cdot)$ is the Airy function.

IV. CONCLUSIONS

As shown in Eq. (27), the solution for far-field waves generated by an oscillating Stokeslet is given by the sum of two wave systems, namely η_S and η_T. As time goes to infinity, η_S persists while η_T eventually vanishes due to the presence of a viscous decay factor $\exp(-2\nu k^2 t)$. Thus, a steady-state wave is attained. Accordingly, η_S can be referred to as the steady-state response while η_T the transient response. It is observed that the steady-state response is a monochromatic wave while the transient response has a dispersive behavior. The transient response consist of two components, namely the long gravity-dominant ($m = 1$) and short capillary-dominant ($m = 2$) waves. The two waves are merged into one at $R/t = C_{\text{gmin}}$, which represents the wave front of the wave systems.

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