SUDDEN ROTATION OF FLOATING PLATE WITH SEPARATION

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1 Introduction

The plane unsteady problem of sudden motion of a floating plate is considered. The plate starts to move at $t' = 0$. The vertical velocity of the plate v_0 and its angular velocity ω_0 are given constants when $t' > 0$. Prime stands for dimensional variables.

Before the impact, $t' < 0$, both the plate and liquid are at rest. The liquid occupies the lower half-plane $y' < 0$. The centre of the plate at $t' = 0$ is taken as the origin of the Cartesian coordinate system $x'Oy'$. The line $y' = 0$ corresponds to the liquid boundary at $t' = 0$. The boundary consists of three intervals. The intervals $-\infty < x' < -L$ and $L < x' < \infty$ are the liquid free surfaces, and the interval $-L < x' < L$ corresponds to a rigid plate of length $2L$, which is floating on the liquid surface with zero draft before the impact. The liquid is assumed ideal and incompressible. Gravity and surface tension effects are not taken into account. The flow generated by the sudden motion of plate motion is two-dimensional and potential.

We assume that the angular velocity of the plate ω_0 is big enough so that a part of the plate exits water due to the plate rotation. Wetted part of the plate is unknown in advance and should be determined as part of the solution. We assume that the liquid is in contact with the plate then and there when and where the pressure is greater or equal to the atmospheric pressure p_{atm}. This condition implies that any force between the liquid particles and the surface of the plate are neglected. In reality, such forces might be rather strong and, correspondingly, the actual wetted area of the plate can be larger than that predicted within the above assumption. We expect that this assumption about the restriction on the pressure in the contact region is valid at least for hydrophobic surfaces. If a potential flow is described by velocity potential $\varphi(x, y, t)$, it can be shown that $\nabla^2[p_{\text{atm}} - p] = \rho[\varphi_{xx}^2 + \varphi_{yy}^2 + 2\varphi_{xy}^2] \geq 0$ in the flow region. This inequality implies that the function $p_{\text{atm}} - p(x, y, t)$ is subharmonic and its maximum cannot be achieved in the interior of the flow region. Therefore, a minimum of the hydrodynamic pressure is achieved at the boundary of the flow region. If the pressure is greater or equal to the atmospheric pressure along the wetted part of the plate, then $p(x, y, t) \geq p_{\text{atm}}$ everywhere in the flow region.

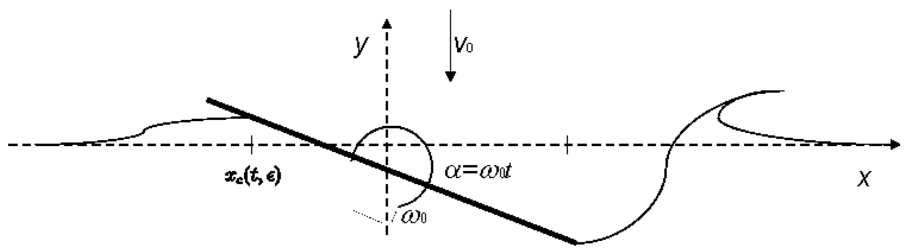

Fig. 1. The flow pattern during the initial stage of the plate rotation.

We shall determine the liquid flow, the pressure distribution along the wetted part of the plate, the length of the wetted part of the plate as a function of time, the total hydrodynamic force and hydrodynamic moment acting on the plate during the initial stage, when both the plate and liquid displacements are small compared with the plate length. The initial velocity field of the flow and initial position of the wetted part of the plate just after the plate starts to move are described by the pressure-impulse theory. The solution provided by this theory is used in this study to formulate the initial conditions for the unsteady problem of floating plate rotation. In particular, we are concerned with the motion of the separation point along the plate during the early stage of the process.

2 Formulation of the problem

The two-dimensional unsteady problem of liquid flow generated by sudden motion of a floating plate is formulated in non-dimensional variables with L being the length scale, $\omega_0 L^2$ the scale of the velocity potential and stream function, ϵ/ω_0 the time scale, where ϵ is a formal small parameter indicating that we consider the initial stage of the flow only, ϵL the displacement scale, and $\rho\omega_0^2 L^2 \epsilon^{-1}$ the pressure scale. Note that scaling is based on the angular velocity of the plate ω_0 but not on its vertical velocity v_0. This implies, in particular, that pure rotation of the plate with $v_0 = 0$ can also be considered. The sketch of the flow is shown in Fig. 1.

The problem is formulated with respect to the complex potential $w(z,t) = \varphi(x,y,t) + i\psi(x,y,t)$, where $z = x + iy$ and $\psi(x,y,t)$ is the stream function. The complex potential $w(z,t)$ is an analytic function in the flow region. The boundary condition on the wetted surface of the moving plate can be integrated and written as

$$\psi = \gamma x + \frac{x^2}{2\cos^2(\epsilon t)} + C(t), \tag{1}$$

where ϵt is the angle of plate rotation and $\gamma = \frac{v_0}{\omega_0 L}$. The function $C(t)$ in (1) has to be determined as part of the solution. The boundary condition (1) is imposed on the wetted part of the moving plate

$$y = -\epsilon t\left[\gamma + x\frac{\tan(\epsilon t)}{\epsilon t}\right], \quad x_c(t,\epsilon) < x < \cos(\epsilon t).$$

Here $x_c(t,\epsilon)$ is the x-coordinate of the separation point. The function $x_c(t,\epsilon)$ describes the length of the wetted part of the rotating plate. This function is unknown and should be determined with the help of additional conditions specifying behaviour of the flow and pressure distribution in a vicinity of the separation point.

The boundary conditions on the free surface of the flow region in non-dimensional variables are

$$\varphi_t + \frac{1}{2}\epsilon|\nabla\varphi|^2 + æ\,\epsilon^2\,\hat{\eta}(x,t,\epsilon) = 0, \qquad \varphi_y = \epsilon\hat{\eta}_x\varphi_x + \hat{\eta}_t, \tag{2}$$

where $æ = g/(\omega_0^2 L)$, $y = \epsilon\hat{\eta}(x,t,\epsilon)$, $x < x_c(t,\epsilon)$ and $x > \cos(\epsilon t)$.

We need to find the complex potential $w(z,t)$ which satisfies the boundary conditions (1) and (2), decays at the infinity, and describes the flow such that the free surface is attached tangentially to the surface of the plate both at the right-hand edge of the plate and at the separation point, and the pressure is positive in the wetted area of the plate.

Due to sudden motion of the plate, the flow just after the impact, $t = +0$, is different from that before the impact, $t = -0$. In order to derive the initial conditions at $t = +0$, we set $\epsilon = 0$ in the boundary conditions (1) and (2) and integrate the dynamic boundary condition in (2) with respect to time. We arrive at the boundary value problem for the complex potential $w_0(z) = w(z,+0)$ in the lower half-plane $y < 0$ with the following boundary conditions

$$\psi_0(x,0) = \gamma x + \frac{1}{2}x^2 + C_0 \qquad (x_{c0} < x < 1),$$

$$\varphi_0(x,0) = 0 \qquad (x < x_{c0}, \;\; x > 1)$$

and conditions that $\psi_0(x,0)$ and $\varphi_0(x,0)$ are continuous functions, $w_0(z) \to 0$ as $|z| \to \infty$, and $\varphi_0(x,0) \le 0$ in the wetted part of the plate. The latter inequality implies that the pressure impulse is positive in the contact region between the liquid and the plate surface. By using the theory of analytic functions and the Hilbert formulas, the velocity potential $\varphi_0(x,0)$ along the wetted part of the plate was found as

$$\varphi_0(x,0) = -\frac{1}{2}(x - x_{c0})^{\frac{3}{2}}(1-x)^{\frac{1}{2}}, \tag{3}$$

where $x_{c0} = -\frac{1}{3}[4\gamma + 1]$. This solution was derived by Norkin in [1] by the method of dual integral equations. It is seen that x_{c0} is always negative and $x_{c0} > -1$, this means there is a separation of the free surface from the plate, only if $\omega_0 > 2v_0/L$. If $v_0 = 0$, then $\gamma = 0$ and $x_{c0} = -\frac{1}{3}$.

3 Dynamics of the separation point

The leading order solution (3) provides initial position of the separation point x_{c0} and the initial velocity field of the flow but does not describe dynamics of the separation point. If the separation point moves significantly after the impact, then this sudden change of the plate wetted area may give an important contribution to the hydrodynamic loads through the higher-order approximations. It is possible that the second-order contribution to the velocity of the flow as $\epsilon \to 0$ has the same order as the displacement of the separation point along the plate. If so, the position of the separation point and the second-order corrections to both the velocity field and pressure distribution should be determined simultaneously.

In order to perform the coupled analysis of the problem, we introduce stretched coordinates ξ and η:

$$x = A(t,\epsilon)\xi + B(t,\epsilon), \quad y = A(t,\epsilon)\eta,$$

$$A = 1 - \frac{1-\cos(\epsilon t)}{1-x_{c0}} - \frac{x_c(t,\epsilon) - x_{c0}}{1-x_{c0}}, \quad B = \frac{1-\cos(\epsilon t)}{1-x_{c0}} x_{c0} + \frac{x_c(t,\epsilon) - x_{c0}}{1-x_{c0}}$$

such that the horizontal coordinates of the wetted part of the plate are fixed in time as $\xi = 1$ and $\xi = x_{c0}$. Correspondingly, new unknown functions in the stretched variables are introduced as

$$\varphi(A\xi + B, A\eta, t, \epsilon) = A(t,\epsilon)\phi(\xi,\eta,t,\epsilon), \qquad \psi(A\xi + B, A\eta, t, \epsilon) = A(t,\epsilon)\chi(\xi,\eta,t,\epsilon)$$

$$\hat{\eta}(A\xi + B, t, \epsilon) = A(t,\epsilon)H(\xi,t,\epsilon)$$

The boundary conditions (1) and (2) in new variables are

$$\chi = \gamma\xi + \frac{1}{2}\xi^2 + c_0 + \frac{1}{2}\xi^2\left(\frac{A(t,\epsilon)}{\cos^2(\epsilon t)} - 1\right) + \frac{B}{A}\left(\gamma + \frac{A(t,\epsilon)}{\cos^2(\epsilon t)}\xi\right) + S\left(\eta = -\epsilon t\left[\frac{\gamma}{A} + (\xi + \frac{B}{A})\frac{\tan(\epsilon t)}{\epsilon t}\right], x_{c0} < \xi < 1\right),$$

$$(A\phi)'_t + \frac{1}{2}\epsilon|\nabla\phi|^2 + \epsilon^2 æ AH - \phi_\xi(A'_t\xi + B'_t) + \epsilon A'_t H\phi_\eta, \;\; \phi_\eta = \epsilon H_\xi \phi_\xi + (AH)'_t - H_\xi(A'_t\xi + B'_t) \quad (\eta = \epsilon H, \xi > 1, \; \xi < x_{c0}).$$

The boundary conditions are more complicated than those in the original variables. However, the stretched variables are more suitable for asymptotic analysis as $\epsilon \to 0$. Here $B \to 0$ and $A \to 1$ as $\epsilon \to 0$. This means that the stretched variables are identical to the original ones in the leading order as $\epsilon \to 0$.

The leading order solution (3) should be considered as the "outer" solution because it is not valid close to the right edge of the plate, where it predicts square-root singularity of the flow velocity. In order to derive a uniformly valid leading-order solution, we need to consider the local flow at this edge of the plate in detail. The inner variables are introduced as

$$\xi = 1 + \epsilon^k\lambda, \;\; \eta = \epsilon^k\mu, \;\; \phi = \epsilon^{k/2}\phi^{(in)}(\lambda,\mu,t,\epsilon), \;\; H = \epsilon^m S(\lambda,t,\epsilon).$$

It is shown that $k = \frac{2}{3}$, $m = -\frac{1}{3}$ and the inner solution in the leading order as $\epsilon \to 0$ is nonlinear and self-similar with respect to the variables

$$\alpha = \frac{\lambda}{(bt)^{\frac{2}{3}}}, \;\; \beta = \frac{\mu}{(bt)^{\frac{2}{3}}}, \;\; \phi_0^{(in)} = at^{\frac{1}{3}}\Phi(\alpha,\beta), \;\; S_0 = (bt)^{\frac{2}{3}}Q(\alpha),$$

where $a = (3D^4/2)^{\frac{1}{3}}$, $b = 3D/2$ and $D = \frac{1}{2}(1-x_{c0})^{\frac{2}{3}}$. The boundary value problem for the inner potential $\Phi(\alpha,\beta)$ has been shown to be identical to the inner problem studied in [2] for the flow generated by sudden vertical motion of a floating plate. By using the results from [2], we conclude that

$$\Phi = r^{\frac{1}{2}}\sin(\theta/2) - C_1 r^{-\frac{1}{2}}\sin(\theta/2) + O(1/r) \quad (r = \sqrt{\alpha^2+\beta^2} \to \infty), \tag{4}$$

where $C_1 \approx -0.41$. Asymptotic formula (4) shows that the velocity potential in the outer region has to behave as

$$\phi^{(outer)}(\xi,0,t,\epsilon) \sim -\frac{1}{2}(1-x_{c0})^{\frac{3}{2}}\sqrt{1-\xi} - \frac{2|C_1|b^{\frac{5}{3}}}{3\sqrt{1-\xi}}(\epsilon t)^{\frac{2}{3}} \quad (\xi \to 1-0) \tag{5}$$

when approaching the plate edge. This estimate and the boundary conditions on the plate imply that the second-order outer velocity potential and the position of the separation point can be approximated as

$$\phi^{(outer)}(\xi,\eta,t,\epsilon) = \varphi_0(\xi,\eta) + \epsilon^{\frac{2}{3}}\phi_1(\xi,\eta,t) + o(\epsilon^{\frac{2}{3}}), \;\; x_c(t,\epsilon) = x_{c0} + \epsilon^{\frac{2}{3}}X_1(t)(1-x_{c0}) + o(\epsilon^{\frac{2}{3}})$$

where $\epsilon \to 0$. It is shown that $\phi_1(\xi,\eta,t)$ is proportional to $t^{\frac{2}{3}}$ and $X_1(t) = -Et^{\frac{2}{3}}X^*$, where X^* is a coefficient to be determined and $E = \frac{2}{3}|C_1|b^{\frac{5}{3}}$.

We introduce the complex potential $W(\zeta)$:

$$w_1(\zeta,t) = -Et^{\frac{2}{3}}W(\zeta),\ \ \zeta = \xi + i\eta,\ \ w_1 = \phi_1 + i\chi_1,\ \ W = \hat{\Phi} + i\hat{\Psi}.$$

The potential $W(\zeta)$ is analytic in $\eta < 0$ and subject to the mixed boundary conditions

$$Re[W(\xi - i0)] = 0 \ \ (\xi > 1 \text{ and } \xi < x_{c0}), \tag{6}$$

$$Im[W(\xi - i0)] = X^*\left(-\frac{1}{2}\xi^2 + \xi + c_1\right) \ \ (x_{c0} < \xi < 1), \tag{7}$$

where X^* and c_1 are undetermined constants. The matching condition (5) gives

$$\hat{\Phi}(\xi,-0) \sim -(1-\xi)^{-\frac{1}{2}} \quad \text{as} \quad \xi \to 1-0. \tag{8}$$

The condition (8) implies that the second-order velocity potential in the main flow region is singular at the right edge of the moving plate. In order to find this singular solution, we introduce the analytic in $\eta < 0$ function

$$V(\zeta) = \frac{i}{(1-x_{c0})^{\frac{3}{2}}}\left(\zeta - \frac{(\zeta - x_{c0})^{\frac{3}{2}}}{(\zeta-1)^{\frac{1}{2}}}\right),$$

which satisfies conditions (6) and (8) and decays at the infinity. However, the imaginary part $Im[V(\xi-0)] = \xi/(1-x_{c0})^{\frac{3}{2}}$, where $x_{c0} < \xi < 1$, does not satisfy condition (7). We introduce new complex potential $W_a(\zeta) = W(\zeta) - V(\zeta)$, which is regular at both ends of the wetted area, and formulate the mixed boundary problem for $W_a(\zeta)$. It is shown that the solution of this problem exists if and only if

$$X^* = \frac{4}{3(1-x_{c0})^{\frac{5}{2}}}.$$

Correspondingly, the second-order approximation of the coordinate of the separation point is given in the non-dimensional variables as

$$x_c(t,\epsilon) = x_{c0} - \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{3}}|C_1|(1-x_{c0})(\epsilon t)^{\frac{2}{3}} + ...$$

and the leading order pressure distribution along the wetted part of the plate as

$$p = 0.15 \times \rho\omega_0(v_0 + \omega_0 L)(x - x_c(t,\epsilon))\frac{\sqrt{x - x_c}}{\sqrt{L-x}}(\omega_0 t)^{-\frac{1}{3}},$$

where approximately $\frac{1}{3}\left(\frac{4}{3}\right)^{\frac{1}{3}}|C_1| \approx 0.15$.

4 Conclusion

The results of the second-order asymptotic analysis with respect to small duration of the initial stage of the plate motion demonstrate that the hydrodynamic pressure just after the impact is dependent on the motion of the separation point. It was shown that the separation occurs if the angular velocity of the plate is relatively large. If the plate starts to rotate suddenly, then the initial position of the separation point is given by the pressure-impulse theory. After the impact the wetted area starts to shrink at velocity which behaves as $O(t^{-\frac{1}{3}})$ as $t \to 0$. The hydrodynamic pressure at the initial stage is of the order of $O(t^{-\frac{1}{3}})$ inside the wetted part of the plate and is equal to the atmospheric pressure at the separation point.

5 References

1. Norkin M.V., (2007) *Mixed boundary problems of hydrodynamic impact*, Rostov-on-Don, South Federal University, 135pp.

2. Iafrati A., Korobkin A.A. (2004) Initial stage of flat plate impact onto liquid free surface. Physics of Fluids. Vol. 16. No. 7, pp. 1214-1227.