A New Algorithm for the Time-domain Green Function

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1. Introduction

In the time domain solution of the three-dimensional unsteady ship motions, the computation of the time-domain Green function which is involved in the convolution integral accounts for the largest portion of the CPU time. In order to solve the realistic ship-wave interaction problem, accurate and fast techniques of evaluating the time domain Green function are needed.

Methods based on the series and asymptotic expansions of the time-domain Green function were presented by Beck and Liapis (1986) and Newman (1992). Later, taking advantage of the fact that the integral to compute may be converted to the function of only two parameters, a bivariate interpolation in a table precomputed once for all and stored in a permanent file was used to evaluate the Green function (Magee and Beck, 1989; Ferrant, 1990). The table interpolation technique allows a substantial cut in the CPU time, but is less accurate than the series expansions and usually requires a large memory space to load the file of the table.

Different from the above widely used evaluation method from the original integral form, the ordinary differential equations (ODE) for the time domain Green function and its spatial derivatives were derived by Clement (1997), and thus the computation of the integral of the oscillatory kernel of the time-domain Green function can be replaced by the integration of differential equations. The ODEs for the Green function are simple and easy to be implemented in the Earth-fixed reference frame. However, these ODEs are complex and not convenient to be used in the moving reference frame, e.g. a reference frame moving with the constant forward speed of ship.

In this paper, a new efficient method of evaluating the time-domain Green function is proposed through the utilization of Dawson integral and Taylor expansions. For illustration, this new algorithm is applied to determine the hydrodynamic forces on a submerged sphere undergoing large-amplitude motions and the submerged sphere with constant forward speed.

2. Computation of the time-domain Green function

In the fluid of infinite depth ,the time-domain Green function is defined by the expression

$$G(P,Q,t) = G^{(0)}(P,Q)\delta(t) + H(t)\tilde{G}(P,Q,t) \tag{1}$$

with

$$\mathbf{G}^{(0)}(\mathbf{P},\mathbf{Q}) = \frac{\mathbf{1}}{\mathbf{r}} - \frac{\mathbf{1}}{\mathbf{r}_1} \tag{2}$$

$$\tilde{G}(P,Q,t) = \mathbf{2}\int_{\mathbf{0}}^{\infty} \sqrt{gk}\,\mathbf{sin}(\sqrt{gk}t)e^{k(z+\zeta)}J_{\mathbf{0}}(kR)dk \tag{3}$$

where $\delta(\mathbf{t})$ is the Dirac impulse, H(t) the Heaviside step function, and J_0 a Bessel function of the first kind of order zero; P(x,y,z) denotes the field point and $Q(\xi,\eta,\zeta)$ the source point, both lying in the lower half space ($z \leqslant 0$, $\zeta \leqslant 0$), $\mathbf{R}=\sqrt{(\mathbf{x}-\xi)^2+(\mathbf{y}-\eta)^2}$, $\overline{\mathbf{Q}}=(\xi,\eta,-\zeta)$, $\mathbf{r}=\left|\mathbf{PQ}\right|$, and $\mathbf{r}_1=\left|\mathbf{P}\overline{\mathbf{Q}}\right|$.

In ship hydrodynamics, $G^{(0)}$ is often referred to as the instantaneous part of the Green function, while $\tilde{G}$ is called the memory part. In this paper our attention will be focused on the memory part, for the sake of brevity, it will be referred to as the Green function hereafter.

As shown in Newman (1992), by a change of variable in Eq. (3), the Green function can be expressed as a function of two real variables (μ, τ)

$$\tilde{G}(P,Q,t)=\sqrt{g}r_1^{-3/2}\,\mathbf{F}(\mu,\tau) \tag{4}$$

with

$$F(\mu,\tau)=2\int_0^\infty \sqrt{\lambda}\,\mathbf{sin}(\sqrt{\lambda}\tau)e^{-\lambda\mu}J_0(\lambda\sqrt{1-\mu^2})d\lambda \tag{5}$$

where $\mu=-(\mathbf{z}+\zeta)/\mathbf{r}_1$ and $\tau=\mathbf{t}\sqrt{\mathbf{g}/\mathbf{r}_1}$.

The integral expression of Bessel function $J_0(k\sqrt{1-\mu^2})$ is

$$\begin{aligned}\mathbf{J}_0(\lambda\sqrt{1-\mu^2})&=\frac{1}{\pi}\int_0^\pi \mathbf{cos}(\lambda\sqrt{1-\mu^2}\,\mathbf{cos}\,\theta)\mathbf{d}\theta\\&=\frac{2}{\pi}\int_0^{\pi/2}\mathbf{cos}(\lambda\sqrt{1-\mu^2}\,\mathbf{cos}\,\theta)\mathbf{d}\theta\end{aligned} \tag{6}$$

(Abramowitz and Stegun, 1972, eq. 9.1.18). Replacing the Bessel function J_0 by its integral representation in eq. (6) and reversing the orders of integration gives

$$\mathbf{F}(\mu,\tau)=\frac{4}{\pi}\int_0^{\pi/2}\int_0^\infty\sqrt{\lambda}\sin(\sqrt{\lambda}\tau)\mathbf{e}^{-\lambda\mu}\cos(\lambda\sqrt{1-\mu^2}\cos\theta)\mathbf{d}\lambda\mathbf{d}\theta \tag{7}$$

Further, the Green function can be expressed in terms of the Dawson integral

$$\begin{aligned}\mathbf{F}(\mu,\tau)=\frac{4}{\pi}\int_0^{\pi/2}\mathrm{Re}\Big(&\frac{1}{2a_1^3}\mathbf{DAW}(\frac{\mathbf{t}}{2\mathbf{a}_1})+\frac{\mathbf{t}}{4a_1^4}-\frac{\mathbf{t}^2}{4a_1^5}\mathbf{DAW}(\frac{\mathbf{t}}{2\mathbf{a}_1})\\&+\frac{1}{2a_2^3}\mathbf{DAW}(\frac{\mathbf{t}}{2\mathbf{a}_2})+\frac{\mathbf{t}}{4a_2^4}-\frac{\mathbf{t}^2}{4a_2^5}\mathbf{DAW}(\frac{\mathbf{t}}{2\mathbf{a}_2})\Big)\,d\theta\end{aligned} \tag{8}$$

where

$$\mathbf{DAW}(\mathbf{z})=\mathbf{e}^{-\mathbf{z}^2}\int_0^{\mathbf{z}}\mathbf{e}^{\mathbf{t}^2}\mathbf{dt}$$

is the Dawson integral(Abramowitz and Stegun, 1972, eq. 7.4.7), a_1 and a_2 are the complex arguments defined as follows

$$\mathbf{a}_1^2=\mu-\mathbf{i}\cos\theta\sqrt{\mathbf{1}-\mu^2}\,,\quad \mathbf{a}_2^2=\mu+\mathbf{i}\cos\theta\sqrt{1-\mu^2}$$

Eq. (8) can be calculated using the composite Simpson's rule to assure a good precision. Efficient algorithms for evaluating the Dawson integral of a complex variable were given by Gautchi (1970), Poppe and Wijers(1990).

When both the field and source points are close to the free surface, we expand the integrand of (5) making use of the Taylor expansion of $e^{-\lambda\mu}$

$$\mathbf{F}(\mu,\tau) = \mathbf{2}\int_{\mathbf{0}}^{\infty} \sqrt{\lambda}\sin(\sqrt{\lambda}\tau)[\mathbf{1} - \lambda\mu + \frac{(\lambda\mu)^{2}}{\mathbf{2!}} - \frac{(\lambda\mu)^{3}}{\mathbf{3!}} + \mathbf{O}((\lambda\mu)^{4})]\mathbf{J}_{\mathbf{0}}(\lambda\sqrt{\mathbf{1}-\mu^{2}})\mathbf{d}\lambda \tag{9}$$

$$= \mathbf{F}_{\mathbf{0}} - \mu\mathbf{F}_{\mathbf{1}} + \frac{\mu^{2}}{\mathbf{2!}}\mathbf{F}_{\mathbf{2}} - \frac{\mu^{3}}{\mathbf{3!}}\mathbf{F}_{\mathbf{3}} + \mathbf{O}(\mu^{4})$$

where,

$$\mathbf{F}_0 = \frac{\pi\omega^3}{\sqrt{2}(1-\mu^2)^{3/2}}[\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{1}{4}}(\frac{\omega^2}{2}) + \mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2})]$$

$$\mathbf{F}_1 = \frac{\pi\omega^2}{\sqrt{2}(1-\mu^2)\tau}\{-\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})_{-\frac{1}{4}}(\frac{\omega^2}{2}) + 3\omega^2[\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2}) - \mathbf{J}_{-\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2})] + 2\omega^4[\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})_{-\frac{1}{4}}(\frac{\omega^2}{2}) + \mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2})]\}$$

$$\mathbf{F}_2 = \frac{-\pi\omega^3}{4\sqrt{2}(1-\mu^2)^{7/4}}\{(25-4\omega^4)\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{1}{4}}(\frac{\omega^2}{2}) + \mathbf{(9\text{-}4}\omega^4\mathbf{)}\mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2}) - 20\omega^2[\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2}) - \mathbf{J}_{-\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2})]\}$$

$$\mathbf{F}_3 = \frac{\pi}{8\sqrt{2}(1-\mu^2)^{9/4}}\{(25\omega - 254\omega^5 + 8\omega^9)\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{1}{4}}(\frac{\omega^2}{2}) + \mathbf{(\text{-}178}\omega^5 + \mathbf{8}\omega^9\mathbf{)}\mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2}) + \mathbf{(\text{-}119}\omega^3 + 84\omega^7\mathbf{)}[\mathbf{J}_{\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{\frac{3}{4}}(\frac{\omega^2}{2}) - \mathbf{J}_{-\frac{1}{4}}(\frac{\omega^2}{2})\mathbf{J}_{-\frac{3}{4}}(\frac{\omega^2}{2})]\}$$

with $\omega = \mathbf{(1}-\mu^{2}\mathbf{)}^{-\frac{1}{4}} \cdot \frac{\tau}{\mathbf{2}}$.

For the case μ=1, the Bessel function J_0 is equal to 1 and the Green function reduces to the following form in terms of the Dawson integral

$$\mathbf{F}(\mathbf{1},\tau) = \mathbf{2}\int_{\mathbf{0}}^{\infty}\sqrt{\lambda}\sin(\sqrt{\lambda}\tau)\mathbf{e}^{-\lambda}\mathbf{d}\lambda = \mathbf{2DAW}(\frac{\tau}{\mathbf{2}}) + \tau - \tau^{2}\mathbf{DAW}(\frac{\tau}{\mathbf{2}}) \tag{10}$$

The algorithms which are well behaved for all real and positive values of τ is given by Codk, Pacirek and Thacher (1970).

For the derivatives of the Green function, formulas similar to (8) and (9) and (10) can also be derived. It can be seen that the present method is easy to be performed in the time-domain analysis.

3. Numerical Results

The function F(μ, τ) is plotted on Figure 1. As μ approaches 0, the oscillatory behavior of F(μ, τ) is amplified.

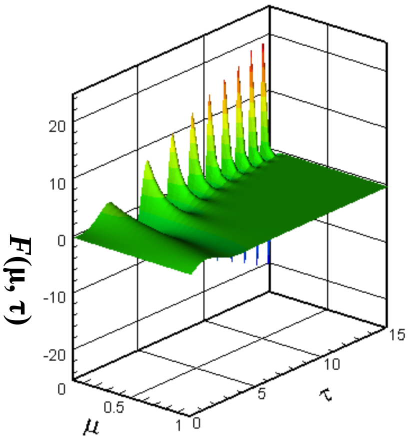

Figure 1. The Green function F(μ, τ)

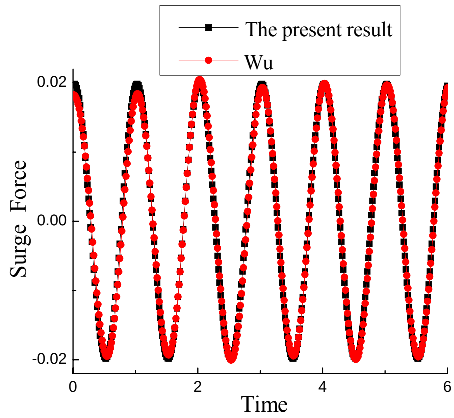

Figure 2. Surge force at Froude number Fr=0.4 and wave number kR=1.0

The surge force on a submerged sphere of the radius R is computed with the present algorithm by setting zero value for m terms. The computed surge force as shown in the Figure 2 agrees well with the analytical solutions by Wu (Chen et al, 2000).

The results on the derivatives of the Green function and the hydrodynamic forces on a submerged sphere undergoing large-amplitude motions will be presented on the workshop.

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