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Visco-potential flow and time-harmonic ship waves

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By analyzing the linear incompressible NS equations through the joint Fourier-Laplace transform, we develop the new formulation combining the kinematic and dynamic conditions over the free surface with viscous terms and surface tension for the linear potential flow, following the recent work by Dutykh & Dias (2007) and Dias *et al.* (2008). Applications of the new boundary condition at the free surface to an elementary wave confirm the decay factor in time same as that in Lamb (1932) and the spatial decay factor. The application of this visco-potential formulation to time-harmonic ship waves generated by a pulsating and advancing point source confirms the results obtained in Chen & Lu (2009). Some new asymptotic analyses are performed, following Chen (2005a), and show that the wave integrals include the exponential decay function representing the dissipation effect of fluid viscosity.

1. Linear incompressible NS equations

We consider the lower half-space filled with water limited on the top by the water-air interface. A Cartesian coordinate system is defined by placing the (x, y)-plane coincided with the undisturbed free surface and the z-axis oriented positively upward. In this gravity-dominant fluid domain, as in Chen & Lu (2009), the reference length L, the acceleration of gravity g and the water density ρ are used to define the nondimensional coordinates $\mathbf{x} = (x, y, z)$, the time t, the fluid velocity $\mathbf{u} = (u, v, w)$, the velocity potential Φ, and the hydrodynamic pressure P with respect to $(L, \sqrt{g/L}, \sqrt{gL}, \sqrt{gL^3}, \rho gL)$ respectively.

By assuming the incompressibility, the fluid flow is governed by the continuity equation to guarantee the conservation of mass :

$$\nabla \cdot \mathbf{u} = 0 \tag{1a}$$

and the momentum equation expressing the conservation of momentum :

$$\mathbf{u}_t = -\nabla P + \epsilon \nabla^2 \mathbf{u} \tag{1b}$$

where $\epsilon = \mu/(\rho\sqrt{gL^3})$ with μ the fluid viscosity. The hydrodynamic pressure P in (1b) includes the dynamic part p and static part z, i.e. $P = p + z$.

On the free surface $z = \eta(x, y, t)$, the boundary conditions are linearized by assuming small wave amplitudes and written on the undisturbed free surface $z = 0$:

$$\eta_t = w \tag{2a}$$

as the kinematic condition stating no fluid particles across the free surface and

$$\epsilon(u_z + w_x) = 0 = \epsilon(v_z + w_y) \tag{2b}$$

$$P = p_a + \eta - \sigma^2(\eta_{xx} + \eta_{yy}) + 2\epsilon w_z \tag{2c}$$

as the dynamic conditions representing the vanishing of shear stress in both x and y directions (2b) and the equation of normal stress (2c). In (2c), p_a stands for the atmospheric pressure, and $\sigma = \sqrt{T/(\rho gL^2)}$ with T the surface tension of water-air interface. In addition, an initial condition can be prescribed for the velocity, the hydrodynamic pressure and the free-surface elevation.

Now, the velocity $\mathbf{u}$ is decomposed as the sum of an irrotational and a solenoidal vectors :

$$\mathbf{u} = \nabla\Phi + \mathbf{u}^T \tag{3}$$

where the scalar function $\Phi(\mathbf{x}, t)$ represents the irrotational flow while $\mathbf{u}^T = (u^T, v^T, w^T)$ the rotational flow. Introducing (3) into the continuous equation and momentum equation (1), we have :

$$\nabla^2\Phi = 0 = \nabla \cdot \mathbf{u}^T \quad \text{and} \quad \mathbf{u}^T_t = \epsilon\nabla^2\mathbf{u}^T \tag{4}$$

The hydrodynamic pressure P is defined by :

$$P = -\Phi_t \tag{5}$$

The boundary conditions (2) can now be expressed in terms of $(\Phi, \mathbf{u}^T)$:

$$\eta_t - (\Phi_z + w^T) = 0 \tag{6a}$$

$$2\Phi_{zx} + u_z^T + w_x^T = 0 = 2\Phi_{zy} + v_z^T + w_y^T \tag{6b}$$

$$\Phi_t + \eta - \sigma^2(\eta_{xx} + \eta_{yy}) + 2\epsilon(\Phi_{zz} + w_z^T) = 0 \tag{6c}$$

on $z = 0$. In (6c) derived from (2c), the atmospheric pressure $p_a = 0$ is taken.

2. Kinematic and dynamic conditions on the free surface including viscosity

Following the work in Dutykh & Dias (2007) and in Dias *et al.* (2008) but with some slight simplifications here, the joint integral transform composed of the Fourier integral with respect to (x, y) and the Laplace transform with respect to t is applied for $(\eta, \Phi, \mathbf{u}^T)$ as in Chen & Lu (2009) :

$$[\tilde{\eta}, \tilde{\Phi}, \tilde{\mathbf{u}}^T] = \int_0^\infty dt \int_{-\infty}^\infty dy \int_{-\infty}^\infty dx \, [\eta, \Phi_0 e^{kz}, \mathbf{u}_0^T e^{k_\epsilon z}] e^{-i(\alpha x + \beta y) - st} \tag{7}$$

in which, we have used the notations :

$$\Phi_0 = \Phi(x, y, z{=}0, t) \quad \text{and} \quad \mathbf{u}_0^T = \mathbf{u}^T(x, y, z{=}0, t)$$

and to satisfy the equations (4), it follows that

$$k = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad k_\epsilon = \sqrt{s/\epsilon + k^2}$$

Taking the joint integral transform (7) over both sides of (6b) as well as $\nabla \cdot \mathbf{u}^T = 0$, we have :

$$2ik\alpha\tilde{\Phi} + k_\epsilon \tilde{u}^T + i\alpha\tilde{w}^T = 0 = 2ik\beta\tilde{\Phi} + k_\epsilon \tilde{v}^T + i\beta\tilde{w}^T \quad \text{and} \quad i\alpha\tilde{u}^T + i\beta\tilde{v}^T + k_\epsilon \tilde{w}^T = 0 \tag{8a}$$

which yields :

$$2k^3\tilde{\Phi} = -(k_\epsilon^2 + k^2)\tilde{w}^T = -(2k^2 + s/\epsilon)\tilde{w}^T \tag{8b}$$

In the same way, taking the joint integral transform (7) to the equation (6a), we have :

$$s\tilde{\eta} - k\tilde{\Phi} - \tilde{w}^T = 0 \tag{8c}$$

Introducing (8b) to (8c), we get :

$$\tilde{w}^T = -2\epsilon k^2 \tilde{\eta} \tag{8d}$$

whose inverse Fourier-Laplace transform gives

$$w^T = 2\epsilon(\eta_{xx} + \eta_{yy}) \tag{9}$$

Introducing (9) to (6a), we obtain a new kinematic condition at $z = 0$:

$$\eta_t - \Phi_z - 2\epsilon(\eta_{xx} + \eta_{yy}) = 0 \tag{10}$$

Since $w = O(\epsilon)$ according to (9), we can rewrite the dynamic condition (6c) as :

$$\Phi_t + \eta - \sigma^2(\eta_{xx} + \eta_{yy}) + 2\epsilon\Phi_{zz} = 0 \tag{11}$$

with an error of order $O(\epsilon^2)$.

3. New combined boundary condition of free-surface flows

As we are interested in the irrotational flow represented by the velocity potential Φ, the Laplace equation (4), the kinematic condition (10) and the dynamic condition (11) define the potential flow with the effect of fluid viscosity. Furthermore, the kinematic (10) and dynamic (11) conditions can be combined together to form a *new* one for Φ :

$$\Phi_{tt} + \Phi_z + \sigma^2\Phi_{zzz} + 4\epsilon\Phi_{zzt} = 0 \tag{12}$$

with an error of order $O(\sigma^4, \sigma^2\epsilon, \epsilon^2)$. For an elementary wave expressed as :

$$\Phi = A\exp[kz + i(kx - \Omega t)] \tag{13a}$$

by assuming k is real (spatially periodic waves). To satisfy the boundary condition (12) at the free surface, we have :

$$\Omega = \omega - 2i\epsilon k^2 + O(\epsilon^2) \quad \text{with} \quad \omega^2 = k + \sigma^2 k^3 \tag{13b}$$

so that

$$\Phi = A_0 e^{-2\epsilon k^2 t}\exp[kz + i(kx - \omega t)] \tag{13c}$$

which confirms that the viscous decay factor $e^{-2\epsilon k^2 t}$ of wave amplitude shown in Lamb (1932) with A_0 the wave amplitude at a reference instant ($t = 0$). Similarly, assuming a time periodic wave with a pure and positive real frequency $\Omega = \omega$, we have :

$$\Phi = A_0 e^{-4\epsilon\omega k^2 x}\exp[kz + i(kx - \omega t)] \tag{13d}$$

in which the factor $e^{-4\epsilon\omega k^2 x}$ indicates the spatial decay along the propagation direction ($x > 0$) and A_0 the wave amplitude at the reference point ($x = 0$).

The notion of fairly perfect fluid by Guével (1982) consists of introducing a dissipation force $-\mu\nabla\Phi$ in the fluid, with a positive real parameter μ, proportional to the magnitude of fluid velocity but in the opposite direction. The Bernoulli equation is then augmented by an additional term $\mu\Phi$. Consequently, the boundary condition at the free surface is added one term $\mu\Phi_t$ as presented in Chen (2004). Comparing with (12), we may find the equivalence if we put $\mu = 4\epsilon k^2$.

4. Time-harmonic ship waves

We consider a point source advancing at a constant speed U along the positive x-direction, and pulsating harmonically at a frequency denoted by ω. The nondimensional frequency f, the Froude number F and the Brard number τ are defined by

$$(f, F, \tau) = (\omega\sqrt{L/g}, U/\sqrt{gL}, U\omega/g) \tag{14}$$

respectively. Under the hypothesis of linear and time-harmonic motions, all physical quantities are expressed by the form $\Phi(\mathbf{x}, t) = \Re\{\phi(\mathbf{x})\exp(-ift)\}$ so that only their complex associate $\phi(\mathbf{x})$ is involved in the following development.

We study the flow in the fluid domain $\mathbf{x} = (a, b, c \le 0)$ due to a point source of the strength equal to 4π located at $\mathbf{x}' = (a', b', c' \le 0)$. By assuming the incompressibility, the fluid flow is governed by the continuity equation :

$$\nabla^2\phi = 4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{15}$$

and the boundary condition at the free surface :

$$(F\partial_x - if)^2\phi + \phi_z + \sigma^2\phi_{zzz} + 4\epsilon(F\partial_x - if)\phi_{zz} = 0 \tag{16}$$

The usual Fourier transform to (16) gives ϕ by the integral representation :

$$\phi = -1/|\mathbf{x}-\mathbf{x}'| + 1/|\mathbf{x}-\mathbf{x}'_1| + \phi^F \tag{17}$$

with $\mathbf{x}'_1 = (a', b', -c')$ is the mirror point of $\mathbf{x}' = (a', b', c')$ with respect to the mean free surface $z = 0$, and ϕ^F is the wave potential function :

$$\pi\phi^F = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \;\; e^{kz+i\varphi}/(D + i\epsilon B) + O(\epsilon^{3/2}) \quad \text{with} \quad \varphi = \alpha x + \beta y \tag{18}$$

in which $(x, y, z) = (a - a',\, b - b',\, c + c')$ while D and B are given by :

$$D = (F\alpha - f)^2 - k - \sigma^2 k^3 \quad \text{and} \quad B = -4(F\alpha - f)k^2 \tag{19}$$

According to the asymptotic analysis in Chen (2005a), the free-surface potential (18) for $x^2 + y^2 \to \infty$ can be approximated by the single integral :

$$\phi^F \approx -i\sum_{D=0}\int_{D=0} ds \left[\operatorname{sign}(B) + \operatorname{sign}(xD_\alpha + yD_\beta)\right]\frac{\mathcal{E}}{\|\nabla D\|}\exp(kz + i\varphi) \tag{20}$$

along the dispersion curves defined by the dispersion relation $D = 0$. In (20), $\sum_{D=0}$ means summation over all the dispersion curves, ds is the differential element of arc length and $\|\nabla D\|^2 = D_\alpha^2 + D_\beta^2$ with $D_\alpha = \partial D/\partial\alpha$ and $D_\beta = \partial D/\partial\beta$. Finally, the term $\mathcal{E}$ is defined as :

$$\mathcal{E} = \exp(-\epsilon B\varphi') \quad \text{with} \quad \varphi' = (xD_\alpha + yD_\beta)/\|\nabla D\|^2 \tag{21}$$

When $\epsilon \to 0$, the term $\mathcal{E} \to 1$ since $B < \infty$ along the dispersion curves. At this limit, (20) becomes Eq.27b in Noblesse & Chen (1995) which represents the wave component. Accordingly, the formula (20) is expected to represent the wave component of (18). Comparing (20) with Eq.27b in Noblesse & Chen (1995) all results obtained in Chen (2005b) on the wave component remain valid. The additional term $\mathcal{E}$ in the amplitude function does not introduce any modification to the wave form. As an exponentially decreasing (damping) term, it represents only the dissipative effect.

5. Discussions and conclusions

To take account of dissipation effect due to fluid viscosity, a modification of Bernoulli's equation and the dynamic condition at the free surface is made classically by adding a dissipation term while only potential component of the velocity is used in the kinematic condition, or the modification of the combined free-surface condition as did in Tuck (1974). Unlike these classical studies, the free-surface flow described by the linear incompressible NS equation and the classical kinematic and dynamic conditions at the free surface are analyzed here through the decomposition (3) of the velocity vector into an irrotational part represented by the gradient of velocity potential and a solendoidal vector, and by making use of the joint Fourier-Laplace transform. The connection between the vertical rotational velocity and the free-surface elevation (9) is found and introduced into the original boundary conditions at the free surface. The resultant kinematic condition (10) and dynamic condition (11) contain each one term proportional to the fluid viscosity which implicitly represents the leading contribution of rotational field. Indeed, the formulations (10) and (11) provide the new kinematic condition and dynamic condition over the free surface taking account of the leading contribution of fluid viscosity. Furthermore, the boundary condition (12) obtained by combining both (10) and (11) is new and particularly interesting since it is satisfied only by the velocity potential and can be applied to a wide class of free-surface flows with the dissipation effect in a direct and handful way.

Applications of (12) to an elementary wave indicate well the decay factor $e^{-2\epsilon k^2 t}$ of wave amplitude in time and $e^{-4\epsilon\omega k^2 x}$ in space. The time-harmonic ship waves generated by a pulsating and advancing source are considered by using the new boundary condition (12). The same expression (18) of the free-surface potential as that in Chen & Lu (2009) is then obtained and associated with the same dispersion relations (19). The asymptotic analysis given in Chen (2005a) is applied to obtain the far-field approximation wave term represented by the single integrals (20) along the dispersion curves, in which the dissipative effect of fluid viscosity is explicitly expressed by the exponential decay function (21).

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