Inertia and Damping of Heaving Compound Cylinders

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We wish to convey our very best wishes to Professor Eatock Taylor on the occasion of his retirement. He has made impressive contributions in a broad range of subjects represented by this conference.

1 Introduction

Since the initial work of Yeung (1981), who solved the radiation problem of a truncated cylinder oscillating with 3 DOF in finite-depth water, there have already been numerous exciting extensions of the method of matched eigenfunction in a variety of related context. Among these studies were interacting multiple cylinders (Yeung & Sphaier, 1989, Yilmaz, 1998), concentric cylinders and cylinder with a moonpool (Mavrako, 1988, 2004, Shipway & Evans, 2003), cylinders in a small current (Kinoshita & Bao, 1996), cylinders with a flexural ice sheet (Malenica & Korobkin, 2003), and details of Helmholtz resonance of twin rectangular cylinders (Seah & Yeung, 2006). The list is not exhaustive. Nevertheless, there is renewed interest in having a fast and reliable way of estimating radiation properties for a compound cylinder. This resurgence of interest derived not so much from offshore engineering, as the original solution was intended, but from various means to extract ocean wave energy. In particular, in a point-absorber design (Yeung et al., 2010), the requirement of a body to have more than one components, with each having one type of motion relative to the other, has led to the need to compute the hydrodynamic properties of each individual section, taking into account the interference effects. To this aim, a careful mathematical analysis similar to Yeung (1981) has been developed to understand the effects of geometric variations on these coefficients, which govern the resonance properties of the point-absorber device. Applications aside, the behavior of this added mass and damping in the low-frequency regime have some intrinsic mathematical interest.

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2 Division of domain & solution forms

The formulation here follows closely of that used by Yeung (1981) and Seah and Yeung (2006). Figure 1 shows a sketch of the problem under consideration. Two concentric truncated circular cylinders are heaving with amplitudes ζ_1 and ζ_2 in water of depth h. If an incompressible fluid without viscosity and surface tension is assumed, the velocity potential, which is axisymmetric, can be represented by:

$$\Phi(r, \theta, z, t) = Re[-i\sigma(\zeta_1\phi_1(r, z, t) + \zeta_2\phi_2)e^{-i\sigma t}] \quad (1)$$

where σ is the angular frequency of the heaving motion.

As shown in Fig. 1, for either ϕ_1 or ϕ_2, with only minor adjustments on the body boundary conditions, the fluid region can be divided into three subregions with different types of spatial variation of the potential in each. These potentials are denoted by $\phi^{iq}(q = 1, 2)$ and ϕ^e. ϕ^{i1} and ϕ^{i2} are the potentials underneath the inner and outer cylinders, respectively, while ϕ^e is the potential associated with the infinite fluid region. In each subregion, the usual boundary conditions apply on the concentric cylinders, the fluid bottom, the free surface and the radiation condition at infinity. Separation of variables enable the velocity potential in each subregion to be expressed as infinite series of vertical eigenfunctions as follows. These expansions are constructed to satisfy all boundary conditions except at the matching boundaries, i.e. at $r = a_1$ and $r = a_2$.

In the regions under the concentric cylinders, the

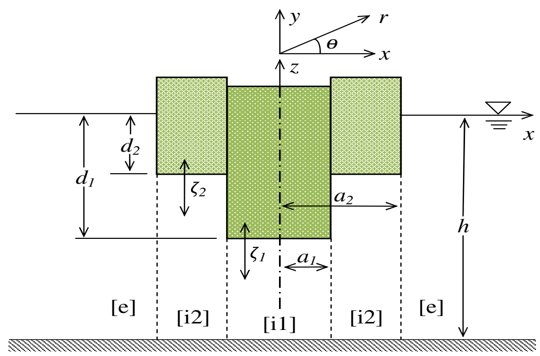


Figure 1: Definition of a compound axisymmetric cylinder.

velocity potentials ϕ^{iq} can be expressed as the sum of a particular (ϕ_p^{iq}) and homogeneous (ϕ_h^{iq}) solution.

$$\phi^{iq} = \phi_p^{iq} + \phi_h^{iq} \tag{2}$$

where ϕ_p^{iq} satisfy the inhomogeneous boundary condition

$$\frac{\partial \phi_p^{iq}}{\partial z} = 1, \quad \text{at } z = -d_q, \tag{3}$$

Note that ϕ_p^{iq} is only needed for inhomogeneous body boundary condition, i.e., for nonzero ζ_i. ϕ_h^{iq} are infinite sets of vertical eigenfunctions with eigenvalues satisfying

$$\lambda_n^q = \frac{n\pi}{h - d_q}, \quad n = 0, 1, \ldots \tag{4}$$

Following Yeung (1981), we can derive, for either $\phi 1$ or $\phi 2$,

$$\phi_p^{iq} = \frac{1}{2(h - d_q)} \left[(z + h)^2 - \frac{r^2}{2} \right] \tag{5}$$

$$\phi_h^{iq} = \sum_{n=0}^{\infty} \left[C_{1n}^q R_{1n}^q(r) + C_{2n}^q R_{2n}^q(r) \right] Z_n^{iq}(z) \tag{6}$$

where C_{1n}^q and C_{2n}^q are unknown coefficients to be determined from conditions at the matching boundaries, i.e. at $r = a_q$. R_{1n}^q, R_{2n}^q and Z_n^{iq} are defined as

$$R_{1n}^q(r) = \begin{cases} 1/2 & \text{for } n = 0 \\ \frac{I_0(\lambda_n^q r)}{I_0(\lambda_n^q a_2)} & \text{for } n \geq 1 \end{cases} \tag{7}$$

$$R_{2n}^q(r) = \begin{cases} 0 & \text{for } q = 1 \\ \frac{1}{2}\ln(r/a_2) & \text{for } n = 0,\ q = 2 \\ \frac{K_0(\lambda_n^q r)}{K_0(\lambda_n^q a_2)} & \text{for } n \geq 1,\ q = 2 \end{cases} \tag{8}$$

$$Z_n^{iq}(z) = \begin{cases} 1 & \text{for } n = 0 \\ \sqrt{2}\cos\lambda_n^q(z + h) & \text{for } n \geq 1 \end{cases} \tag{9}$$

Note that in region [i2] of the concentric cylinder, the modified Bessel function K_0 and a $\ln(r)$ term are required to furnish a complete expansion for the annular cylindrical boundary. Z_n^{iq} are sets of orthonormal functions such that the inner product can be defined as follows for every integer pair l and m

$$< Z_l^{iq}, Z_m^{iq} >= \frac{1}{h - d_q} \int_{-h}^{-d_q} Z_l^{iq}(z) Z_m^{iq}(z) dz = \delta_{lm} \tag{10}$$

For the exterior fluid region, ϕ^e can be expanded into another series of vertical eigenfunction with eigenvalues associated with the linearized free-surface and bottom conditions, i.e.

$$\begin{aligned} m_0 \tanh m_0 h &= \sigma^2/g \\ m_k \tan m_k h &= -\sigma^2/g, \quad k = 1, 2, \ldots \end{aligned} \tag{11}$$

$$\phi^e = \sum_{k=0}^{\infty} B_k \Lambda_k(r) Z_k^e(z) \tag{12}$$

where

$$\Lambda_k(r) = \begin{cases} \frac{H_0^{(1)}(m_0 r)}{H_0^{(1)}(m_0 a_2)} & \text{for } k = 0 \\ \frac{K_0(m_k r)}{K_0(m_k a_2)} & \text{for } k \geq 1 \end{cases} \tag{13}$$

$$Z_k^e(z) = \begin{cases} N_0^{-\frac{1}{2}} \cosh m_0(z + h) & \text{for } k = 0 \\ N_k^{-\frac{1}{2}} \cos m_k(z + h) & \text{for } k \geq 1 \end{cases} \tag{14}$$

and N_k are well-known scale factor defined by Eqn. (16) of Seah & Yeung (2006) such that Z_k^e form an orthonormal set:

$$< Z_k^e, Z_j^e >= \frac{1}{h} \int_{-h}^{0} Z_k^e(z) Z_j^e(z) dz = \delta_{k_j} \tag{15}$$

Again, the unknown coefficients B_k are determined from the matching conditions at boundary $r = a_2$. As a remark, the present formulation can easily be modified to study the hydrodynamic coefficients of a truncated cylinder with a concentric *moonpool*. In this case, $d^{i1} = h$, and the fluid in this inner region is required to satisfy the linearized free-surface boundary condition. ϕ^{i1} can be expressed by the following expansion

$$\phi^{i1} = \sum_{k=0}^{\infty} A_k \Gamma_k(r) Z_k^e(z) \tag{16}$$

where

$$\Gamma_k(r) = \begin{cases} \frac{J_0(m_0 r)}{J_0(m_0 a_1)} & \text{for } k = 0 \\ \frac{I_0(m_k r)}{I_0(m_k a_1)} & \text{for } k \geq 1 \end{cases} \tag{17}$$

and A_k are unknown coefficients to be determined at the matching boundary $r = a_1$

3 Matching of solutions

To determine the values of C_{1n}^q, C_{2n}^q and B_k from infinite series (6) and (12), it is required that the potentials and fluxes be matched along the vertical boundaries of adjacent fluid regions. Together with the kinematic boundary conditions along the vertical surfaces of the concentric cylinders, and the orthonormal properties

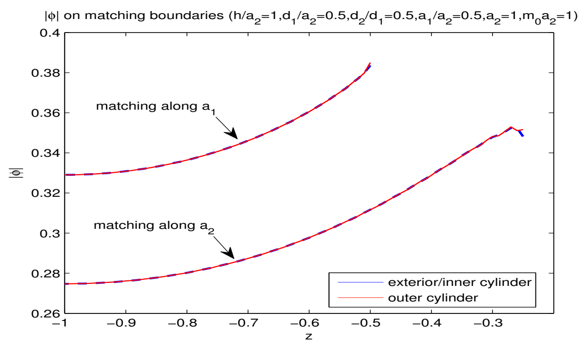
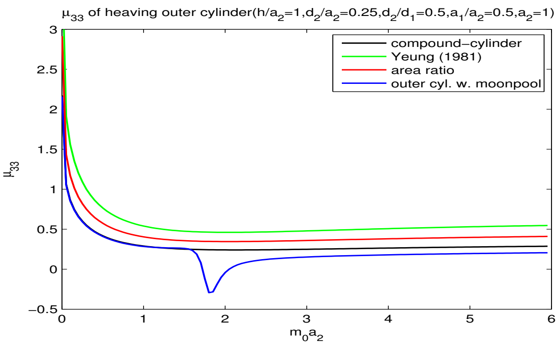
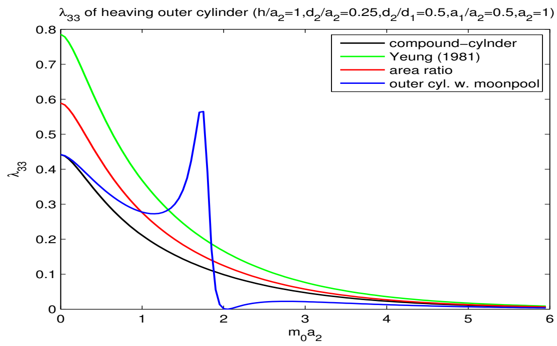


Figure 2: Matching potentials on region boundaries.



(a) μ_{33}



(b) λ_{33}

Figure 3: μ_{33} and λ_{33} of a compound cylinder with heaving outer cylinder and fixed inner cylinder.

of the sets: $[Z_n^{iq}]$ and $[Z_k^e]$, ϕ_i can be expressed entirely in terms of C_{1n}^2 and C_{2n}^2 only. These coefficients can then be solved explicitly from a linear system of equations, after truncating the infinite series to finite number of terms, say, N.

4 Results and Discussions

The results presented here are based on a basic geometric configuration of $a_1/a_2 = 0.5$, $d_2/a_2 = 0.25$, and $d_1/d_2 = 2$ with $N = 50$. Two water depths are considered: $h/a_2 = 1$ and $h/a_2 = 5$. μ_{33} and λ_{33} are nondimensionalized by $\pi\rho a_2^3$ for heaving outer cylinder, and by $\pi\rho a_1^3$ for heaving inner cylinder. Figure 2 illustrates the effectiveness of the matching procedure. $|\phi|$ for adjacent fluid regions are plotted along the matching boundaries, i.e. at $r = a_1$ and $r = a_2$. Generally at $N = 50$, the matching is accurate up to 4 digits except near the corners. The integrable corner singularity was discussed in Yeung and Wu (1989). This singularity is weak and depends on the discontinuous boundary condition at the corner.

Figure 3a and 3b show respectively μ_{33} and λ_{33} of the outer cylinder with heaving motion, while the inner cylinder is fixed. Also, plotted in these figures are the corresponding hydrodynamic coefficients of (a) a single-cylinder with the same radius and draft of the outer cylinder (Yeung, 1981), (b) the compound-cylinder results obtained from a geometric approximation of applying the area factor to the single-cylinder results, and (c) the same outer cylinder though the inner cylinder is replaced by a concentric moonpool (Mavrakos, 1988). It can be observed that if the Helmholtz frequency is identified at the critical frequency ω_0 at which $\lambda_{33} = 0$ (Seah and Yeung, 2006). Interestingly, at frequencies $< \omega_0, \mu_{33}$ of the compound-cylinder results agree well with those of a cylinder with a moonpool, but are smaller at frequencies $> \omega_0$. λ_{33} exhibits similar behavior at frequencies far from ω_0. Surprisingly, the variation of the hydrodynamic coefficients of the compound cylnder resemble closer to those of a cylinder with a moonpool than to those of a single cylinders. It is also of interest to observe from Fig. 4 that the low-frequency limits of λ_{33} for both the compound-cylinder and the cylinder with a moonpool approach the same value, which is numerically different from those of single cylinders in Yeung (1981). This limit depends on water depth and can be shown to be given by $\pi(a_2^2 - a_1^2)^2/(4ha_2^3)$. The derivation of this asymptotics will be presented in the workshop. The above observations encourage us to examine the influence of the fixed inner cylinder on the heaving outer cylinder.

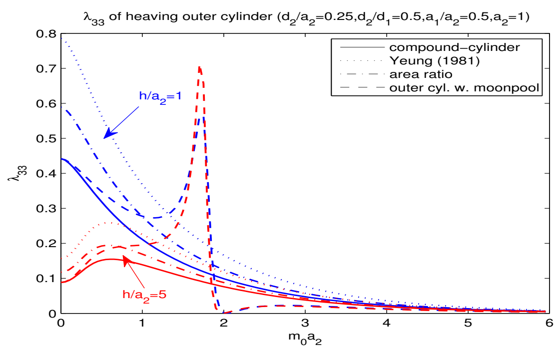
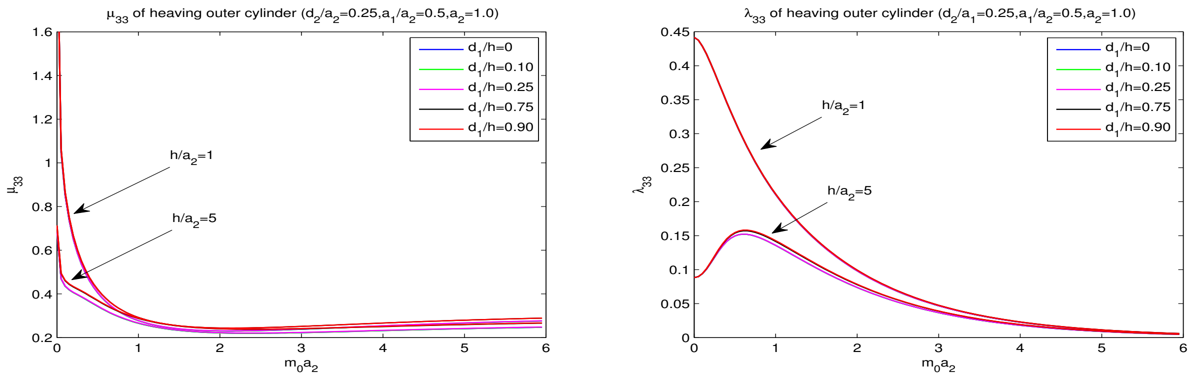


Figure 4: Low frequency behaviors of λ_{33} for the heaving outer cylinder at water depth $h/a_2 = 1$ and $h/a_2 = 5$.

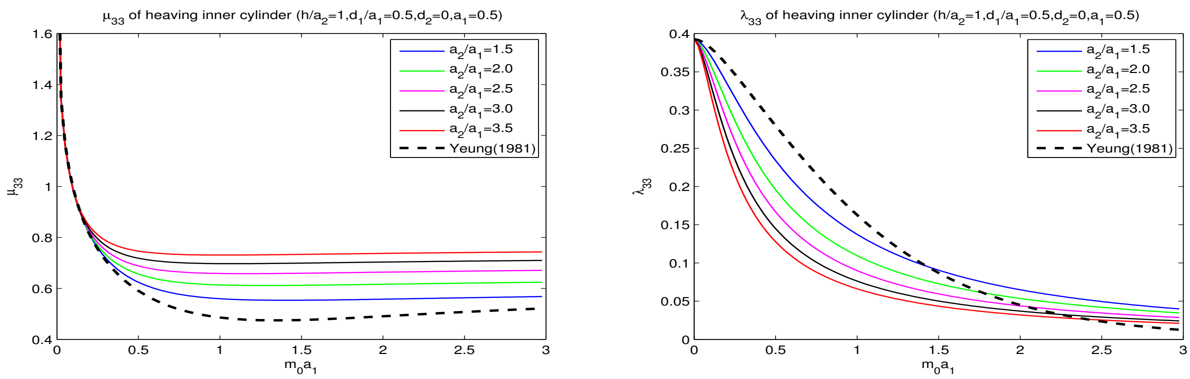
The effects for five different drafts of the inner cylinder, (i.e. $d_1 = 0, 0.1, 0.25, 0.75, 0.9$) are shown in Figs. 5a and 5b. It can be observed that the variations of the draft of the inner cylinder have practically very little effect on μ_{33} and λ_{33} of the outer cylinder. The fluid appears to be moving outwards rather within the "plugged" region.

How then would the outer cylinder affect the coefficients of the inner cylinder? It can be shown that, unlike the previous scenario, the hydrodynamic coefficients of the heaving inner cylinder are strongly influenced by the fixed outer cylinder. In general, μ_{33} increases but λ_{33} decreases when compared with a single cylinder (of the same radius and draft of the inner cylinder) heaving in a laterally unbounded fluid. An interesting limiting case to study this behavior is to



(a) μ_{33}. (b) λ_{33}.

Figure 5: Effect of d_1 on μ_{33} and λ_{33} of the heaving outer cylinder at water depths $h/a_2 = 1$ and $h/a_2 = 5$.



(a) μ_{33}. (b) λ_{33}.

Figure 6: Variations of μ_{33} and λ_{33} of heaving inner cylinder with increasing radius of the outer cylinder at draft $d_2 = 0$.

make the draft of the fixed outer cylinder zero, i.e. a disk. This also minics the effect of a rigid ice sheet surrounding the heaving inner cylinder. Figures 6a and 6b show the results of varying the radius a_2 of these "ice sheets". It is evident that increasing a_2 increases μ_{33} but decreases λ_{33}. This can be explained by knowing that the "ice sheet" confines the fluid flow around the heaving inner cylinder in a vertical region, thus increasing its added mass. The "ice sheet" also limits the wave-making ability of the heaving inner cylinder, hence reducing the damping.

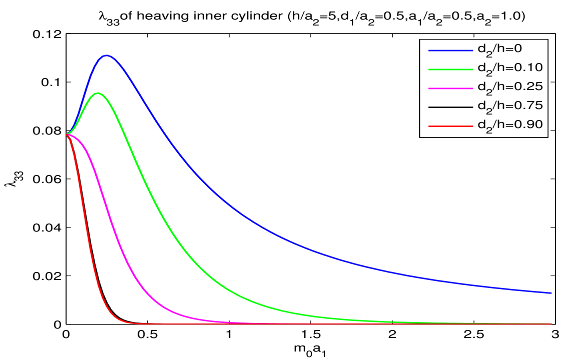


Figure 7: Low frequency behaviors of λ_{33} for the heaving inner cylinder at water depth $h/a_2 = 5$.

It is noteworthy that even in the presence of the outer cylinder, the low-frequency limit of λ_{33} of the heaving inner cylinder approaches the same limit of a single cylinder of Yeung (1981). This low-frequency limit is independent of the drafts of the outer fixed cylinder and is confirmed in Figure 7. It is also observed that λ_{33} decreases with increasing drafts of the outer cylinder.

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