The two dimensional Green's function in elliptic coordinates

by

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1 Introduction

The analytic solution of the hydrodynamic diffraction induced by elliptical cylinders, has admittedly attracted limited attention by researchers and is confined within the frames of the linear theory. Examples of studies reported in the literature that treat only isolated bodies (bottom seated and truncated) are those due to Williams [1-2] and Williams and Darwiche [3]. Recently, the present authors developed an efficient semi-analytical formulation and after reproducing the same results extending their work to solve the hydrodynamic diffraction problem by multiple bottom seated elliptical cylinders [4-5].

The solution of the linear problem gives rise to start considering higher-order problems starting from the double frequency one. It evident that such a task is very demanding and extremely difficult as several problems must be surmounted. One of the crucial problems is the derivation of the Green's function that governs the spatial variation of the "locked wave" component of hydrodynamic diffraction. The present work is devoted to this task and the associated procedure is outlined succinctly in the following.

2 The second-order hydrodynamic problem

Let φ_2 denote the total second-order diffraction potential by an elliptical cylinder. The advisable procedure that has prevailed in the literature is to decompose φ_2 into a "locked wave" component $\varphi_2^{(DD)}$ and a "free wave" component $\varphi_2^{(ID)}$. The "locked wave" component must satisfy the Laplace equation, the kinematical condition on the bottom, the inhomogeneous condition on the free surface, and the homogeneous kinematical condition on the wetted surface of the body, according to which the fluid velocity on the body due to $\varphi_2^{(DD)}$ must be zero. The free surface condition for the second-order problem is written as

$$\left(-4\omega^2/g\,\varphi_2^{(DD)} + \partial\varphi_2^{(DD)}/\partial z\right)_{z=h} = F(u,v) \tag{1}$$

where ω is the incident wave frequency, g is the acceleration due to gravity, h is the water depth and $F(u,v)$ is the free surface pressure distribution due to the first order quadratic components. In elliptic coordinates u and v are intersecting families of confocal ellipses and hyperbolae respectively. An appropriate formulation that satisfies the bottom and the free surface boundary conditions is given by

$$\varphi_2^{(DD)} = \frac{\cosh(\lambda z)}{\omega^2/g\cosh(\lambda h)}F(u,v) + \sum_{j=0}^{\infty} R_j(u,v)Z_j^{(2)}(z) \tag{2}$$

where $\lambda\tanh(\lambda h)=\omega^2/g$ and $Z_j^{(2)}(z)$ are the second-order vertical eigenfunctions [6]. After introducing Eq. (2) into the Laplace equation and using the orthogonality relation of $Z_j^{(2)}(z)$ it can be shown that the "locked wave" component is given by the following compact formula

$$\varphi_2^{(DD)} = \sum_{l=0}^{\infty}\Phi_l(u,v)Z_l^{(2)}(z) \tag{3}$$

where $\Phi_l(u,v)=[Z_l^{(2)}(h)hF(u,v)/(\lambda^2h^2+\sigma_l^2h^2)+R_l(u,v)]$ with σ_l being the solution of the second-order transcendental equation $4\omega^2/g+\sigma_l\tanh(\sigma_l h)=0$. In addition it follows that $\Phi_l(u,v)$ must satisfy the inhomogeneous Helmholtz equation

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$$\nabla^2\Phi_l(u,v) - \sigma_l^2\Phi(u,v) = f(u,v) \tag{4}$$

where $f(u,v)=1/hZ_l^{(2)}(h)F(u,v)$.

3 Two dimensional Green's function in elliptic coordinates

Equation (4) should be treated in 2D space and in particular on the free surface at the area that is defined outside the elliptical body Σ (Fig.1) and extends to infinity. With reference the notations in Fig. 1 we define $R^2=\xi^2+\eta^2$, $\rho^2=x^2+y^2$ which yield $q^2=(x-\xi)2+(y-\eta)^2= \rho^2+R^2-2\rho R\cos(\theta-\vartheta)$. Using Green's theorem, it can be shown [7] that the solution for $\Phi_l(u,v)$ is given by

$$\Phi_l(x,y) = \iint_\Omega f(\xi,\eta)G_l(x,y;\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \tag{5}$$

where $f(\xi, \eta)=1/hZ_l^{(2)}(h)F(\xi, \eta)$ and G_l is the two dimensional Green's function associated with the eigenmode l. Green's function must satisfy the homogeneous Helmoholtz equation, it must be bounded at infinity and it must have a zero first derivative normal to the contour of the body Σ. Green's function must also comply with the following requirements: $\lim_{\rho\to 0}\rho(\partial G_l/\partial\rho)_{q=\rho} = 0$ and $\lim_{\rho\to 0}\rho G_l = 0$. These imply that G_l must behave as ln(q). A function that features these properties and in addition satisfies Helmholtz equation is the zero order Hankel function H_0. Therefore G_l is written as

$$G_l(x,y;\xi,\eta) = \mathrm{i}/4\mathrm{H}_0(\kappa_l q) + \overline{H}_l(x,y;\xi,\eta) \tag{6}$$

where $\kappa_l^2=-\sigma_l^2$, while the last function was put artificially in order to satisfy the kinematical condition on the contour of the body. Next, using Graf's addition theorem Eq. (6) is recast to

$$G_l(x,y;\xi,\eta) = \begin{cases} \mathrm{i}/4\sum_{m=-\infty}^{\infty}\mathrm{J}_m(\kappa_l\rho)\mathrm{H}_m(\kappa_l R)e^{im(\theta-\vartheta)} + \overline{H}_l(x,y;\xi,\eta), & R>\rho \\ \mathrm{i}/4\sum_{m=-\infty}^{\infty}\mathrm{J}_m(\kappa_l R)\mathrm{H}_m(\kappa_l\rho)e^{im(\theta-\vartheta)} + \overline{H}_l(\xi,\eta;x,y), & R<\rho \end{cases} \tag{7}$$

It is now time to apply elliptical coordinates and define $x=c\cosh u\cos v$, $y=c\sinh u\sin v$; $\xi=c\cosh\beta\cos\gamma$, $\eta=c\sinh\beta\sin\gamma$, where c is given in terms of the elliptic eccentricity ε and the semi-major axis of the elliptical cylinder a as $c=a\varepsilon$. Equation (7) is transformed into elliptical coordinates by using the Bessel to Mathieu functions addition theorem [8]:

$$\mathrm{Z}_m^{(j)}(\kappa_l R)e^{\mathrm{i}m\theta} = \sum_{p=-\infty}^{\infty} d'_{p,m}(q_l^{(2)})\mathrm{M}_{m+p}^{(j)}(u;q_l^{(2)})\mathrm{me}_{m+p}(u;q_l^{(2)}) \tag{8}$$

In Eq. (8) j=1 to 4 denotes the kind of the radial Mathieu function $\mathrm{M}^{(j)}$, while $\mathrm{Z}^{(j)}$ denotes respectively for j=1 to 4 the Bessel and Hankel functions J, Y, $\mathrm{H}^{(1)}$ and $\mathrm{H}^{(2)}$. Finally me denotes the periodic Mathieu functions, while $d'_{p,m}$ are coefficients associated with the series expansion coefficients of Mathieu functions [8]. These coefficients depend also on the Mathieu parameter $q_l^{(2)}=\kappa_l^2c^2/4$. Introducing Eq. (8) into Eq. (7) and performing extensive mathematical manipulations the Green's function G_l is recast to elliptical coordinates according to

$$G_l(u,v;\beta,\gamma) = \begin{cases} \mathrm{i}/4\sum_{n=-\infty}^{\infty}(-1)^n\mathrm{B}_{nl}(\beta,\gamma)\mathrm{M}_n^{(1)}(u;q_l^{(2)})\mathrm{me}_n(v;q_l^{(2)}) + \overline{H}_l(u,v;\beta,\gamma), & R>\rho \\ \mathrm{i}/4\sum_{n=-\infty}^{\infty}(-1)^n\mathrm{B}_{nl}(\beta,\gamma)\mathrm{M}_n^{(3)}(u;q_l^{(2)})\mathrm{me}_n(v;q_l^{(2)}) + \overline{H}_l(u,v;\beta,\gamma), & R<\rho \end{cases} \tag{9}$$

where

$$\mathrm{B}_{nl}(\beta,\gamma) = \sum_{s=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}(-1)^s d'_{s,n-s}d'_{p,s-n}\mathrm{M}_{s-n+p}^{(1)}(\beta;q_l^{(2)})\mathrm{me}_{s-n+p}(\gamma;q_l^{(2)}) \tag{10}$$

4 Application of the kinematical boundary condition on the 2D contour of the elliptical cylinder

The relation which must satisfy the associated boundary condition is given in Eq. (6) according to which the Green's function has been decomposed into two parts. The first has been already determined and is expressed in detail in Eqs. (9) and (10). The next step is the derivation of function $\overline{H}_l(u,v;\beta,\gamma)$ which according to the analysis that preceded must satisfy the homogeneous Helmholtz equation and must be bounded at infinity. After employing elliptic coordinates, Helmholtz equation can be written as

$$\frac{\partial^2\overline{H}_l}{\partial u^2} + \frac{\partial^2\overline{H}_l}{\partial v^2} + q_l^{(2)}(\cosh 2u - \cos 2v)\overline{H}_l = 0 \tag{11}$$

Equation (11) is the well-known Mathieu equation, the solution of which is given as the linear superposition of the infinite linearly independent solutions. Thus,

$$\overline{H}_l(u,v;\beta,\gamma)=\sum\nolimits_{n=-\infty}^{\infty}C_{nl}(\beta,\gamma)\mathrm{M}_n^{(3)}(u;q_l^{(2)})\mathrm{me}(v;q_l^{(2)}) \tag{12}$$

It is noted that the radial constituent part of Eq. (11) has two solutions, namely the modified Mathieu functions of the first and the third kind $\mathrm{M}^{(1)}$ and $\mathrm{M}^{(3)}$, respectively. Nevertheless, the former is omitted as it tends to infinity for large arguments u, thus satisfying the already set requirement for a bounded solution at infinity. The unknown at the moment coefficients $C_{nl}(\beta,\gamma)$ will be calculated by applying the kinematical condition on Σ which in elliptic coordinates is expressed as

$$\left(\partial G_l/\partial u\right)_{u=u_0}=0 \tag{13}$$

The final form for Green's function is obtained after introducing Eqs. (9), (10) and (12) into the boundary condition (13) and performing several mathematical manipulations. This is written as

$$G_l(u,v;\beta,\gamma)=\begin{cases} \mathrm{i}/4\sum\limits_{n=-\infty}^{\infty}\sum\limits_{s=-\infty}^{\infty}\sum\limits_{p=-\infty}^{\infty}(-1)^{n+s}d'_{s,n-s}d'_{p,s-n}\mathrm{me}_n(v;q_l^{(2)})\mathrm{me}_{s-n+p}(\gamma;q_l^{(2)})\mathrm{M}^{(3)}_{s-n+p}(\beta;q_l^{(2)})\cdot \\ \cdot\left[\mathrm{M}_n^{(1)}(u;q_l^{(2)})-\dfrac{\mathrm{M}_n'^{(1)}(u_0;q_l^{(2)})}{\mathrm{M}_n'^{(3)}(u_0;q_l^{(2)})}\mathrm{M}_n^{(3)}(u;q_l^{(2)})\right], \quad R>\rho \\ \mathrm{i}/4\sum\limits_{n=-\infty}^{\infty}\sum\limits_{s=-\infty}^{\infty}\sum\limits_{p=-\infty}^{\infty}(-1)^{n+s}d'_{s,n-s}d'_{p,s-n}\mathrm{me}_n(v;q_l^{(2)})\mathrm{me}_{s-n+p}(\gamma;q_l^{(2)})\mathrm{M}^{(3)}_{n}(u;q_l^{(2)})\cdot \\ \cdot\left[\mathrm{M}_{s-n+p}^{(1)}(\beta;q_l^{(2)})-\dfrac{\mathrm{M}_n'^{(1)}(u_0;q_l^{(2)})}{\mathrm{M}_n'^{(3)}(u_0;q_l^{(2)})}\mathrm{M}_{s-n+p}^{(3)}(u;q_l^{(2)})\right], \quad R<\rho \end{cases} \tag{14}$$

where the primes here denote differentiation with respect to the argument u.

5 Numerical results

Figures 2-5 in the following provide a visual representation on how Green's function behaves on the free surface. The associated results correspond to an ellipse with a/b=0.4, where b denotes the semi-minor axis and u_0=0.423. The wave frequency corresponds to ka=1.0 where k is the wave number. Green's function is given as a function of u and v assuming β=1.224 and γ=0°. Two modes were considered, i.e. the imaginary solution (l=0) and the first real root (l=1) of the second-order transcendental equation. Apparently, the variation of Green's function for l=0 is smoother. In both cases Green's function decays progressively for large arguments, while for l=1 the decay is more abrupt. It is also important to note that at the singular point $u=\beta$, $v=\gamma$, Green's function admits a maximum as expected. In fact, after this point for $u>\beta$, Green's function decreases promptly. Finally, an important characteristic which must be highlighted and cannot be ascertained by the theoretical elaboration is the symmetry of Green's function on either side of the x-z plane.

6 References

[1] Williams AN. Wave forces on an elliptic cylinder. Journal of Waterways, Port, Coastal Ocean Div. American Society of Civil Engineers (1985a), 111: 433-449.

[2] Williams AN. Wave diffraction by elliptical breakwaters in shallow water, Ocean Engineering (1985b), 12: 25-43.

[3] Williams AN, Darwiche MK. Three dimensional wave scattering by elliptical breakwaters, Ocean Engineering (1988), 15: 103-118.

[4] Chatjigeorgiou IK, Mavrakos SA. Hydrodynamic diffraction by multiple elliptical cylinders, Proc 24th Int Workshop for Water Waves and Floating Bodies, Zelonogorsk, Russia (2009): 38-41.

[5] Chatjigeorgiou IK, Mavrakos SA. An analytical approach for the solution of the hydrodynamic diffraction by arrays of elliptical cylinders. Applied Ocean Research (2009), doi:10.1016/j.apor.2009.11.004.

[6] Mavrakos SA, Chatjigeorgiou IK. Second-order diffraction by a bottom seated compound cylinder, Journal of Fluids and Structures (2006), 22: 463-492.

[7] Dettman JW. Mathematical methods in physics and engineering, Dover Publications (1988).
[8] Abramowitz M, Stegun IA. Handbook of mathematical functions. Dover Publications Inc, New York (1970).
[9] Meixner J, Schäfke FW. Mathieusche funktionen und sphäroidfunktionen. Springer, Berlin (1954).

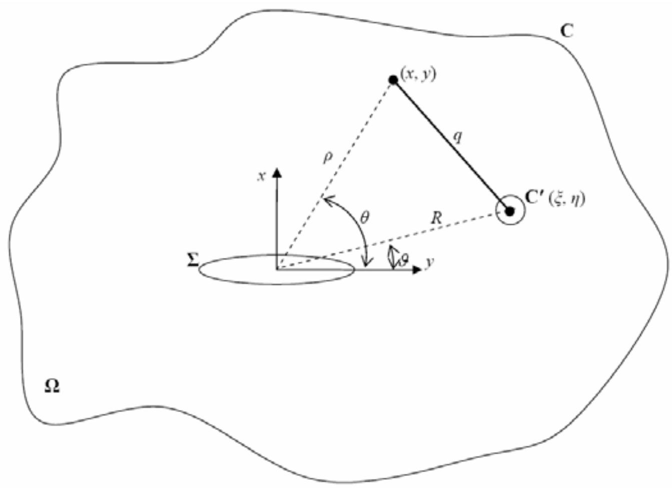

Fig. 1. The reference domain **Ω** extends to infinity **C** and does not contain the elliptical cylinder **Σ**.

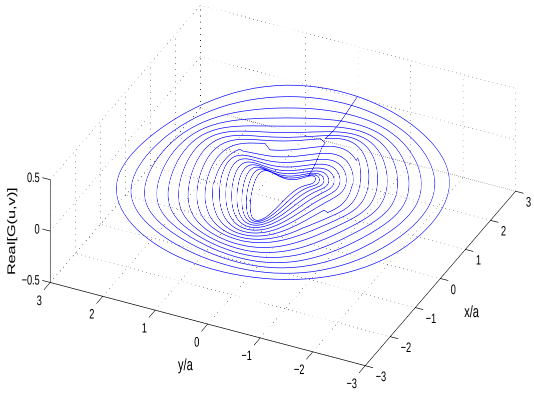

Fig. 2. Real[$G_0(u,v)$] for the imaginary solution

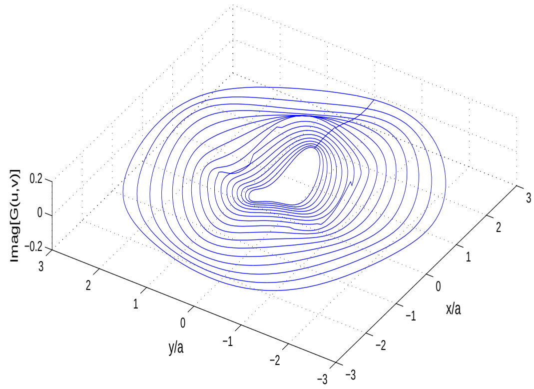

Fig. 3. Imag[$G_0(u,v)$] for the imaginary solution

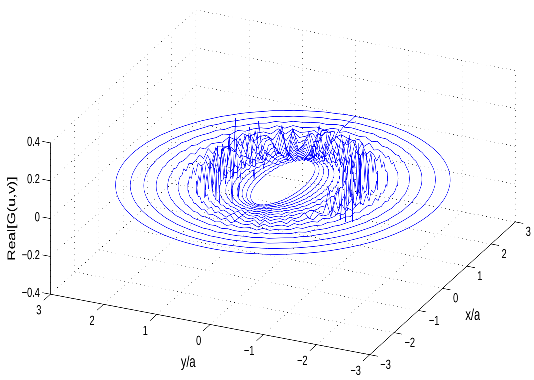

Fig. 4. Real[$G_1(u,v)$] for the first evanescent mode

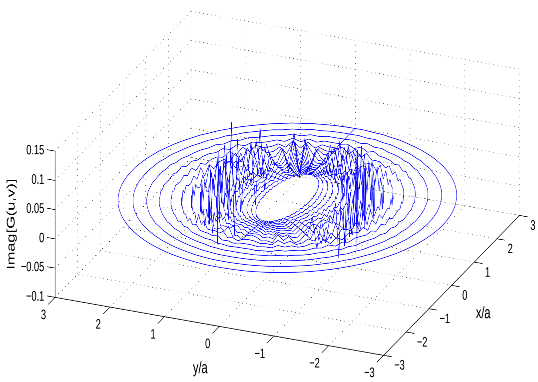

Fig. 5. Imag[$G_1(u,v)$] for the first evanescent mode