# ELASTIC DEFORMATIONS OF A POROUS CIRCULAR CYLINDER FIXED IN WAVES

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## 1. PROBLEM TO BE DEALT WITH

A circular cylinder with a radius $a$, is fixed in water of depth $h$. The cover and bottom of the cylinder are horizontally placed at a depth of $d_1$ and $d_2$ respectively beneath the water surface with $0 < d_1 < d_2 < h$. The height of the cylinder is thus to be $d = d_2 - d_1$. The cylinder is made of porous materials. A train of regular incident waves with an amplitude $A$ and a frequency $\omega$ is propagating in the direction of $\theta = 0$.

As an extension of the previous work[1], the elastic deformation of the cylinder surface is to be investigated within the context of linear hydroelastic theory. The brims of the cylinder's cover and bottom, i.e. $r = a$, $z = -d_1$ and $z = -d_2$, are presumed to be fixed. Therefore, the effects of the rigid-body responses to the incident waves in six degrees of freedom are not considered in the present work.

## 2. NATURAL MODES OF DEFORMATION

The porous cylinder surface is treated as covered by a layer of membrane. Following the conventional expression used in the studies of hydroelasticity, the displacement or the deformation of the cylinder' surface can be expanded in terms of a set of modal functions.

$$w^{(l)} = \mathrm{Re}\left\{ e^{-i\omega t} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \zeta_{np}^{(l)} W_{np}^{(l)} \right\} \quad (l = 1 \sim 3) \tag{1}$$

where $\zeta_{np}^{(l)}$ ($l$ = 1~3) represents the unknown complex modal amplitude corresponding to the $np$-th mode with the superscript ($l$) indicating the cover ($l$ = 1), bottom ($l$ = 2) or side wall ($l$ = 3) of the cylinder respectively. The deformation is assumed to be small so that the displacement can be considered in the normal direction of corresponding surface, i.e. in the $z$–direction for the cover and bottom whilst in the radical direction for the side wall.

The natural modes of membrane vibration are chosen as the modal functions as shown below

$$W_{np}^{(1,2)}(r,\theta) = C_{np}^{(l)} J_n\left(\beta_{np}^{(1,2)} r\right) \cos n\theta \qquad C_{np}^{(l)} = \frac{\sqrt{\varepsilon_n}}{J_{n+1}\left(\beta_{np}^{(1,2)} a\right)} \qquad \varepsilon_n = \begin{cases} 1 & (n = 0) \\ 2 & (n \geq 1) \end{cases}$$
$$W_{np}^{(3)}(\theta, z) = 2\sqrt{\varepsilon_n} \cos n\theta \sin \sqrt{\beta_{np}^{(3)2} - \tfrac{n^2}{a^2}}\,(z + d_2) \tag{2}$$

They are solutions of the free vibration equation for a circular membrane ($l$ = 1, 2) (see Meirovitch) [2] and a cylindrical membrane ($l$ = 3) respectively, i.e.

$$L^{(l)}\{W\} + \beta^2 W = 0; \quad L^{(1,2)}\{W\} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial W}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}; \quad L^{(3)}\{W\} = \frac{1}{a^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial^2 W}{\partial z^2}; \quad \beta^2 = \frac{\overline{\rho}\varpi^2}{T} \tag{3}$$

where $\overline{\rho}$ represents the density of the membrane per unit area, $\varpi$ denotes the natural frequency of free vibration and $T$ stands for the initial tension. The eigen value $\beta$ is determined by substituting the solution into the boundary condition at the fixed brim, i.e. $W^{(1,2)}(a,\theta)=0$ or $W^{(3)}(\theta,-d_s)=0$ ($s$ = 1or 2). Hence, $\beta_{np}^{(1,2)}=x_{np}/a$ with $x_{np}$ denoting the $p$-th root of $J_n$(x), the Bessel function of the first kind, while $\beta_{np}^{(3)}=\sqrt{(n/a)^2+(p\pi/d)^2}$ . The natural mode functions $W_{np}^{(l)}$ are orthogonal and normalized as shown by integral in Eq. (4). Due to the symmetry, natural modes associated with sin $n\theta$ are dropped.

$$\iint_{S_l} W_{np}^{(l)} W_{mq}^{(l)} ds = S_l \delta_{nm} \delta_{pq} \qquad S_{1,2}=\pi a^2 \quad S_3 = 2\pi a d \tag{4}$$

## 3. VELOCITY POTENTIALS

As the fluid motion is considered, the fluid is assumed to be inviscid and the flow to be irrotational. The velocity potential $\Phi(\boldsymbol{x}, t)$ is decomposed in the following way.

$$\Phi(\boldsymbol{x},t)=\mathrm{Re}\left\{\left[\frac{gA}{i\omega}\phi_D(\boldsymbol{x})-i\omega\sum_{l=1}^{3}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\zeta_{np}^{(l)}\phi_{np}^{(l)}(\boldsymbol{x})\right]e^{-i\omega t}\right\} \tag{5}$$

In Eq. (5), $g$ designates the gravitational acceleration. The diffraction potential is solved in the previous work[1]. The detailed discussion will be referred there. It will be concentrated on the solution of radiation potential corresponding to each mode of deflections in the present work. First, the boundary condition on the surface of the cylinder is considered. It is noted that the cylinder is made of porous materials. The Darcy's law is applied on the body surface. Hence, the boundary condition at the cover or bottom of cylinder may be written as:

$$\left.\frac{\partial\phi_{np}^{(l)}}{\partial z}\right|_{(z=-d_s+0)}=\left.\frac{\partial\phi_{np}^{(l)}}{\partial z}\right|_{(z=-d_s-0)}=W_{np}^{(l)}\delta_{ls}+i\sigma_s\left[\left.\phi_{np}^{(l)}\right|_{(z=-d_s-0)}-\left.\phi_{np}^{(l)}\right|_{(z=-d_s+0)}\right] \quad (s=1,\ 2; l=1\sim 3) \tag{6}$$

where $\delta_{ls}$ denotes the Kroenecker function, i.e $\delta_{ls}$ = 1 as $l = s$ or 0 otherwise. On the sidewall of cylinder, the boundary condition is stated as:

$$\left.\frac{\partial\phi_{np}^{(l)}}{\partial r}\right|_{(r=a+0)}=\left.\frac{\partial\phi_{np}^{(l)}}{\partial r}\right|_{(r=a-0)}=W_{np}^{(l)}\delta_{l3}+i\sigma_3\left[\left.\phi_{np}^{(l)}\right|_{(r=a-0)}-\left.\phi_{np}^{(l)}\right|_{(r=a+0)})\right] \quad (l=1\sim 3) \tag{7}$$

In the boundary conditions (6) and (7), the mode function $W_{np}^{(l)}$ represents the normal velocity of the body caused by the displacement of membrane deflection.

Then the radiation potentials are solved in a similar way as the diffraction potential, i.e. by means of eigen function expansions. The fluid domain is divided into two regions, i.e. an exterior one I defined by $r > a$ and an interior one II as $r < a$. Different eigen functions are used in these two regions.

The solution of radiation potential valid in the exterior region is expressed as below.

$$\phi_{np}^{(l)}(\boldsymbol{x})=\cos n\theta\sum_{m=0}^{\infty}A_m^{(lnp)}R_n(k_m r)Z_m(z) \tag{8}$$

The eigen functions $R_n(k_m r)$ and $Z_m(z)$ are well known and their definition is omitted here to save space. In the interior region, due to the deformation of the cylinder cover and bottom, there exists a normal velocity $W_{np}^{(l)}$ ($l$ = 1 or 2) in the boundary condition (see Eq. 6). Therefore, a particular solution $\psi_{np}^{(l)}$ is needed to content this normal velocity. The radiation potentials in the interior region are then written in the following form for $l$ =1 and 2.

$$\varphi_{np}^{(l)}(\boldsymbol{x}) = \cos n\theta \left[ \psi_{np}^{(l)} + \sum_{q=1}^{\infty} [B_q^{(l\,np)} J_n(K_q r) / J_n(K_q a)] f(K_q z) \right] \qquad (l = 1 \text{ or } 2) \tag{9}$$

In Eq. (9), the part of summation in $q$ represents the general solution, of which the eigen functions and the complex eigen values $K_q$ are the same as the diffraction potential.

To determine the particular solution $\psi_{np}^{(l)}$, it is noticed that the natural mode function $W_{np}^{(l)}$ itself satisfies a two-dimensional Helmholtz (see Eq. 3). Therefore, only a proper eigen function of $z$ variable is required. Consequently, the particular solution can be expressed in the following form.

$$\psi_{np}^{(l)} \cos n\theta = W_{np}^{(l)}(r,\theta) \bar{f}_l(\beta_{np} z) = C_{np}^{(l)} J_n(\beta_{np}^{(l)} r) \bar{f}_l(\beta_{np}^{(l)} z) \cos n\theta \qquad (l = 1,\ 2) \tag{10}$$

It is easy to derive the boundary value problem governing the function $\bar{f}_l(\beta_{np} z)$ by separation of variables.

$$\frac{\partial^2 \bar{f}}{\partial z^2} - \beta_{np}^{(l)2} \bar{f} = 0 \qquad (-h < z < 0)\,; \qquad \frac{\partial^2 \bar{f}}{\partial z^2} - \ \nu \bar{f}\ = 0 \qquad (z = 0) \quad \nu = \frac{\omega^2}{g}\,;$$

$$\frac{\partial \bar{f}}{\partial z} = 0 \qquad (z = -h)\,; \qquad \left.\frac{\partial \bar{f}}{\partial z}\right|_{z=-d_s+0} = \left.\frac{\partial \bar{f}}{\partial z}\right|_{z=-d_s-0} = \delta_{sl} + i\sigma_s [\bar{f}|_{z=-d_s-0} - \bar{f}|_{z=-d_s+0}] \quad (s = 1,\ 2) \tag{11}$$

This problem is similar to the one satisfied by the eigen function $f(K_q z)$ in the general solution[1]. Nevertheless the real number of parameter $\beta_{np}^{(l)}$ is known while the complex eigen value $K_q$ should be determined by a complicated dispersion relation. The solution of (11) can be expressed as below.

$$\bar{f}_l(\beta_{np} z) = \begin{cases} P_{np}^l \sinh \beta_{np}(h - d_2) \overline{D}(\beta_{np} z) & (-d_1 < z < 0) \\ \sinh \beta_{np}(h - d_2) \left[ P_{np}^l \overline{D}(\beta_{np} z) + Q_{np}^l \cosh \beta_{np}(z + d_1) \right] & (-d_2 < z < -d_1) \\ \left( P_{np}^l D(\beta_{np} d_2) - Q_{np}^l \sinh \beta_{np}(d_2 - d_1) \right) \cosh \beta_{np}(z + h) & (-h < z < -d_2) \end{cases} \tag{12}$$

It is easy to verify that the governing equation and boundary conditions given in Eq. (11) are satisfied except the last one, i.e. the body surface condition. When the above solution is inserted into this boundary condition, it yields a system of linear algebraic equations. The coefficients $P_{np}^l$ and $Q_{np}^l$ are thus determined and their tedious expressions are omitted here.

For the radiation potential corresponding to the deflection of side wall ($l$ =3), the particular solution is not necessary since the normal velocity at the cover or bottom vanishes.

The unknown coefficients $A_m^{(l\,np)}$ and $B_q^{(l\,np)}$ appeared in the expansion (8) and (9) can be determined by matching these two solutions at the common surface $r = a$.

## 4. MOTION EQUATION AND HYDRODYNAMICE FORCES

The deformation of the cylinder surface is governed by the following motion equation for the forced oscillation of membrane. (see Meirovitch)[2].

$$\bar{\rho}\frac{\partial^2 w^{(l)}}{\partial t^2} - TL^{(l)}\left\{w^{(l)}\right\} = \mathrm{Re}\left\{\Delta p_l(r,\theta)e^{-i\omega t}\right\} \quad (l = 1 \sim 3) \tag{13}$$

The right hand side, i.e. $\Delta p_l(r,\ \theta)$ represents the difference of hydrodynamic pressure on two sides of membrane. Substituting the expansion of $w^{(l)}$ given in Eq. (1), the motion equation can be rewritten as:

$$\sum_{m=0}^{\infty}\sum_{q=1}^{\infty}\zeta_{mq}^{(l)}\left[-\omega^2\bar{\rho}W_{mq}^{(l)} - TL^{(l)}\left\{W_{mq}^{(l)}\right\}\right] = \Delta p_l(r,\theta) \quad (l = 1 \sim 3) \tag{14}$$

Then, both sides of Eq. (14) are multiplied by a mode function $W_{np}^{(l)}$ and integrated over the surface $S_l$ of cover, bottom or side wall as $l$ =1, 2 or 3 respectively. The results can be expressed in the following form.

$$\sum_{m=0}^{\infty}\sum_{q=1}^{\infty}\left\{\zeta_{mq}^{(l)}\left[C_{npmq}^{(l)} - \omega^2 M_{npmq}^{(l)}\right] + \sum_{s=1}^{3}\zeta_{mq}^{(s)}\left[-\omega^2\lambda_{npmq}^{(l,s)} - i\omega\mu_{npmq}^{(l,s)}\right]\right\} = f_{np}^{(l)} \quad (l = 1 \sim 3) \tag{15}$$

The coefficient $M_{nmst}^{(l)}$ , as an element of mass matrix, is defined by the following integrals:

$$M_{npmq}^{(l)} = \bar{\rho}\iint_{S_l} W_{np}^{(l)}(r,\theta)W_{mq}^{(l)}(r,\theta)ds = M_l\delta_{nm}\delta_{pq} \quad (l = 1, 2) \qquad M_l = \begin{cases} \bar{\rho}\pi a^2 & (l = 1, 2) \\ \bar{\rho}\pi a(d_2 - d_1) & (l = 3) \end{cases} \tag{16}$$

Due to the orthogonal property of mode functions, the mass matrix is diagonal as shown above. The coefficient $C_{npst}^{(l)}$ is the element of stiffness matrix caused by the tension distributed over the membrane, which is also diagonal as expressed below.

$$C_{npmq}^{(l)} = -T\iint_{S_l} L^{(l)}\left\{W_{mq}^{(l)}\right\}W_{np}^{(l)}ds = \beta_{mq}^{(l)2}T\delta_{nm}\delta_{pq} \quad (l = 1 \sim 3) \tag{17}$$

The radiation potential makes its contribution to the hydrodynamic coefficients, i.e. added mass and damping, which can be computed by the integral shown in Eq. (18). Meanwhile, shown in Eq. (18) as well is the wave-exciting force evaluated by the integral of diffraction pressure difference.

$$\lambda_{npmq}^{(l,s)} + i\frac{\mu_{npmq}^{(l,s)}}{\omega} = \rho\int_{S_l}\Delta\phi_{mq}^{(s)}W_{np}^{(l)}ds; \qquad f_{np}^{(l)} = \rho\int_{S_l}\Delta\phi_D W_{np}^{(l)}ds \qquad (l,\ s = 1 \sim 3) \tag{18}$$

where the potential difference in the integrand is defined as follows for different surface $S_l$.

$$\Delta\phi = \begin{cases} \varphi(z = -d_l - 0) - \varphi(z = -d_l + 0) & (l = 1,\ 2) \\ \varphi(r = a - 0) - \phi(r = a + 0) & (l = 3) \end{cases} \tag{19}$$

It can be observed that the deformation of one membrane would affect the deflection of the other one. In other words, the deformations of two layers of membrane are coupled. Solving the final form of motion equation (16), the modal amplitude $\zeta_{st}^{(l)}$ is determined.

## REFERENCES


1. Bao, W.G. and Kinoshita, T. *Wave forces acting on a semisuberged aquacultere fish cage*, Proceedings of 25th IWWWFB, 2008.
2. Meirovitch, L. *Analytical method in vibratios*, Macmillan Company, New York, 1967.