Added mass for wave motion in density-stratified fluids

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In a density-stratified fluid the existence of a restoring force – buoyancy – alters fundamentally the dynamics of submerged bodies. This effect can be described as a modification of their added mass. We investigate it in the simplest possible configuration – the small-amplitude motion of an unbounded Boussinesq uniformly stratified fluid – and for the simplest possible bodies – a horizontal circular cylinder in two dimensions and a sphere in three dimensions.

General case The concept of added mass pertains to the irrotational flow of a homogeneous fluid[1]. It follows from the linearity of the flow, such that the translation of a rigid body at the velocity $\mathbf{U}$ creates a velocity potential $\phi = \phi_i U_i$. By defining the added mass tensor m_{ij} according to

$$m_{ij} = \rho \oint_S n_i \phi_j \, \mathrm{d}^2 S,$$

with ρ the density of the fluid, S the surface of the body and $\mathbf{n}$ the inward normal to S, we may express the pressure force on the body as

$$F_i = \oint_S p n_i \, \mathrm{d}^2 S = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \oint_S \phi n_i \, \mathrm{d}^2 S = -m_{ij} \frac{\mathrm{d}U_j}{\mathrm{d}t},$$

the impulse and (kinetic) energy of the fluid as, respectively,

$$I_i = \rho \oint_S \phi n_i \, \mathrm{d}^2 S = m_{ij} U_j, \qquad E = \frac{1}{2} \rho U_i \oint_S \phi n_i \, \mathrm{d}^2 S = \frac{1}{2} m_{ij} U_i U_j,$$

and the dipole strength of the body through

$$\rho \mathcal{D}_i = \rho \oint_S \left(n_i \phi - x_i \frac{\partial \phi}{\partial n} \right) \mathrm{d}^2 S = (m_\mathrm{f} \delta_{ij} + m_{ij}) U_j,$$

with V the volume of the body and $m_\mathrm{f} = \rho V$ the mass of the displaced fluid. Accordingly, added mass characterizes the flow fully, being involved in the dynamics of the body through $\mathbf{F}$, in the dynamics of the fluid as a whole through $\mathbf{I}$ and E, and in the dynamics of the distant fluid through $\boldsymbol{\mathcal{D}}$.

The small-amplitude Boussinesq motion of a uniformly stratified fluid of buoyancy frequency N can similarly be described[2–4] in terms of a scalar function χ, satisfying the internal wave equation

$$\left(\frac{\partial^2}{\partial t^2} \nabla^2 + N^2 \nabla_\mathrm{h}^2 \right) \chi = 0,$$

with the z-axis directed vertically upwards and the subscript $_\mathrm{h}$ denoting a horizontal projection. The velocity $\mathbf{u}$ and the disturbances p in pressure and ρ in density are related to the wave function through

$$\mathbf{u} = \left(\frac{\partial^2}{\partial t^2} \nabla + N^2 \nabla_\mathrm{h} \right) \chi, \qquad p = -\rho_0 \left(\frac{\partial^2}{\partial t^2} + N^2 \right) \frac{\partial \chi}{\partial t}, \qquad \rho = \rho_0 \frac{N^2}{g} \frac{\partial^2 \chi}{\partial t \partial z},$$

with g the acceleration due to gravity, while the pressure p_0 and density ρ_0 at rest satisfy $\mathrm{d}p_0/\mathrm{d}z = -\rho_0 g$ and $\mathrm{d}\rho_0/\mathrm{d}z = -\rho_0 N^2/g$. As for irrotational flow, the linearity of the wave equation implies a linear relation between the velocity $\mathbf{U}$ of a translating rigid body and the wave function that it creates, in the form of a temporal convolution $\chi = \chi_i * U_i$.

The pressure force on the body follows immediately as

$$F_i = \oint_S p n_i \, \mathrm{d}^2 S = -\rho_0 \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} + N^2\right) \oint_S n_i \chi \, \mathrm{d}^2 S.$$

Equations of conservation may be derived for the momentum and total (kinetic and potential) energy of the fluid, of respective densities $\rho_0 u_i$ and $\frac{1}{2}\rho_0 u_i^2 + \frac{1}{2}\rho_0 N^2 \zeta^2$, with $\zeta = \partial^2\chi/\partial t \partial z$ the vertical displacement of fluid particles. By integrating the associated fluxes over the surface of the body, we obtain the momentum and energy outputs of the body as, respectively,

$$I_i = \rho_0 \oint_S \left(n_i \frac{\partial^2}{\partial t^2} + n_{\mathrm{h}_i} N^2\right) \chi \, \mathrm{d}^2 S, \qquad E = \rho_0 U_i \left(\frac{\partial^2}{\partial t^2} + N^2\right) \oint_S n_i \chi \, \mathrm{d}^2 S.$$

Introduction of the Green's function of the wave equation[5] combined with application of standard techniques[6,7] yields a Kirchhoff–Helmholtz integral equation for internal waves, which corrects [8] and generalizes [9]. By expanding this equation at large distances from the body, we obtain the dipole strength of the body, such that

$$\rho_0 \mathcal{D}_i = \rho_0 \oint_S \left[\left(n_i \frac{\partial^2}{\partial t^2} + n_{\mathrm{h}_i} N^2\right) \chi - x_i \left(\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial n} + N^2 \frac{\partial}{\partial n_{\mathrm{h}}}\right) \chi\right] \mathrm{d}^2 S.$$

Accordingly, two distinct definitions of the added mass tensor may be proposed. One,

$$m_{ij}^{(1)} = \rho_0 \left(\frac{\partial^2}{\partial t^2} + N^2\right) \oint_S n_i \chi_j \, \mathrm{d}^2 S,$$

involved in pressure and energy through

$$F_i = -m_{ij}^{(1)} * \frac{\mathrm{d}U_j}{\mathrm{d}t}, \qquad E = U_i \left[m_{ij}^{(1)} * U_j\right],$$

and the other

$$m_{ij}^{(2)} = \rho_0 \oint_S \left(n_i \frac{\partial^2}{\partial t^2} + n_{\mathrm{h}_i} N^2\right) \chi_j \, \mathrm{d}^2 S,$$

involved in momentum and dipole strength through

$$I_i = m_{ij}^{(2)} * U_j, \qquad \rho_0 \mathcal{D}_i = \left[m_{\mathrm{f}} \delta_{ij} \delta(t) + m_{ij}^{(2)}\right] * U_j.$$

The relation between the two is simplified in the monochromatic case, when the excitation $\mathbf{U}$ and the responses $\mathbf{u}$, p and ρ depend on time through the factor $\mathrm{e}^{-\mathrm{i}\omega t}$ which is suppressed in the following. The time-dependent added masses are replaced by their temporal Fourier transforms and the above relations assume the more familiar forms

$$F_i = \mathrm{i}\omega m_{ij}^{(1)} U_j, \quad \langle E \rangle = \tfrac{1}{2} \operatorname{Re}\left[\overline{U_i} m_{ij}^{(1)} U_j\right], \quad I_i = m_{ij}^{(2)} U_j, \quad \rho_0 \mathcal{D}_i = \left[m_{\mathrm{f}} \delta_{ij} + m_{ij}^{(2)}\right] U_j,$$

where $\bar{}$ denotes a complex conjugate and $\langle\,\rangle$ a time average. The two added mass tensors are related anisotropically through

$$m_{ij}^{(1)} = \left(1 - \frac{N^2}{\omega^2}\delta_{i3}\right) m_{ij}^{(2)}.$$

As did [10–12], in the following we will consider the first definition only and omit the superscript $^{(1)}$.

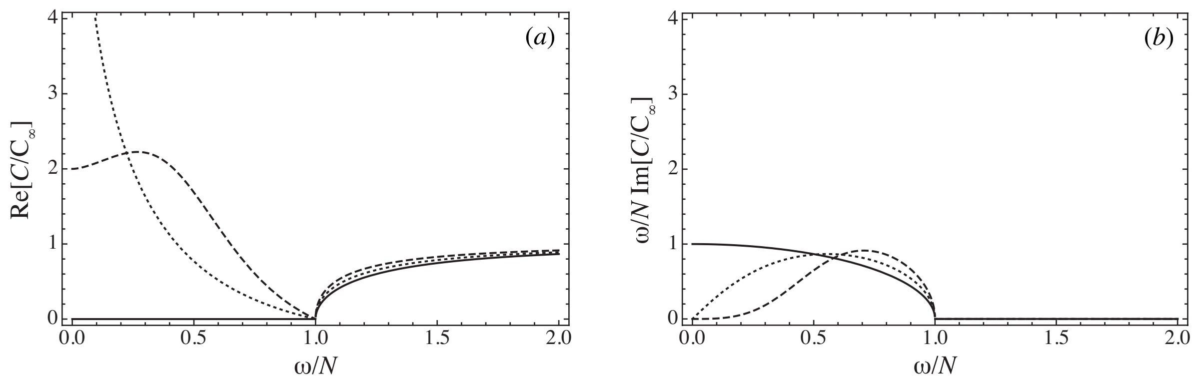

Figure 1 – Coefficients of (a) added mass and (b) damping for an oscillating circular cylinder (———) and a sphere oscillating horizontally (– – – –) or vertically (------).

Oscillating bodies We illustrate these results for two particular oscillating bodies, a horizontal circular cylinder and a sphere, represented as source terms $q = \sigma\delta_S$ on the right-hand side of the wave equation, namely as surface distributions of singularities of density σ. The condition of fixed normal velocity U_n on S becomes an integral equation for σ, solved by use of stretched orthogonal curvilinear coordinates[13] and expansion into circular or spherical harmonics[14]. For rigid oscillation at the velocity $\mathbf{U}$ the solution is found for the cylinder, of radius a, as

$$q(\mathbf{x}) = \left\{\left[1 + \left(1 - \frac{N^2}{\omega^2}\right)^{1/2}\right] U_x \frac{x}{a} + \left[1 + \left(1 - \frac{N^2}{\omega^2}\right)^{-1/2}\right] U_z \frac{z}{a}\right\} \delta(r-a),$$

with $r = |\mathbf{x}|$, and for the sphere, also of radius a, as

$$q(\mathbf{x}) = \left[\frac{2}{1 + B(\omega/N)} \mathbf{U}_\mathrm{h} \cdot \frac{\mathbf{x}_\mathrm{h}}{a} + \frac{1}{1 - B(\omega/N)} U_z \frac{z}{a}\right] \delta(r-a),$$

with

$$B\left(\frac{\omega}{N}\right) = \frac{\omega^2}{N^2}\left[1 - \left(\frac{\omega^2}{N^2} - 1\right)^{1/2} \arcsin\left(\frac{N}{\omega}\right)\right].$$

Complex square roots are fixed, in accordance with causality, by replacing ω by $\omega + \mathrm{i}0$, in other words by adding to the real frequency ω a positive imaginary part which is later allowed to tend to zero.

The associated added mass coefficients $C_{ij} = m_{ij}/(\rho_0 V)$, such that $\mathcal{D}_i = \int x_i q(\mathbf{x})\,\mathrm{d}^3x = V[\delta_{ij} + C_{ij}/(1 - \delta_{i3}N^2/\omega^2)]U_j$, are complex. Their real part represents added inertia and their imaginary part, only present when $\omega < N$ namely when propagative waves are generated, represents wave damping. They are given for the cylinder by

$$C_\mathrm{h} = C_z = \left(1 - \frac{N^2}{\omega^2}\right)^{1/2},$$

and for the sphere by

$$C_\mathrm{h} = \frac{1 - B(\omega/N)}{1 + B(\omega/N)}, \qquad C_z = \left(1 - \frac{N^2}{\omega^2}\right) \frac{B(\omega/N)}{1 - B(\omega/N)}.$$

Their variations, consistent with direct calculations[10,12,15,16] and measurements[17], are represented in figure 1. In the limit $\omega/N \to \infty$ the values $C_\infty = 1$ for the cylinder and $\frac{1}{2}$ for the sphere in a homogeneous fluid are recovered.

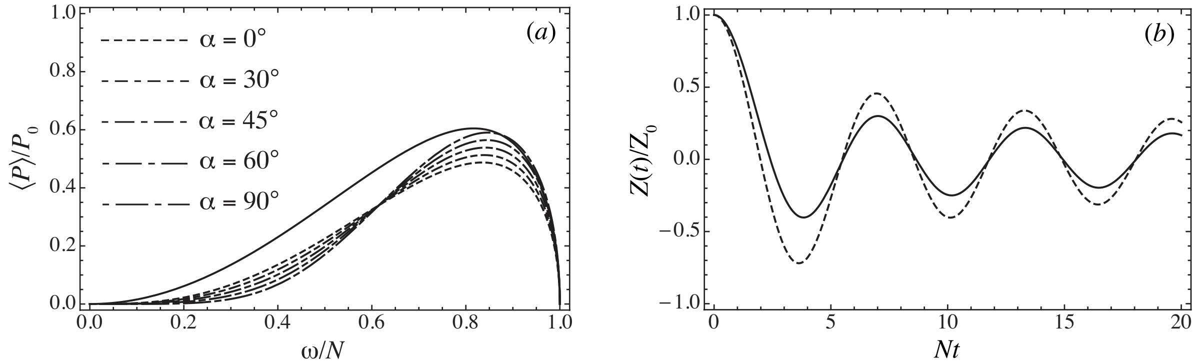

Figure 2 – For a circular cylinder (continuous line) and a sphere (dashed lines), (*a*) power output of forced oscillations, normalized by $P_0 = \rho_0 N^3 a^2 A^2$ for the cylinder and $\rho_0 N^3 a^3 A^2$ for the sphere, and (*b*) free buoyant oscillations.

Two applications A first application of the above is the average power output $\langle P \rangle$ of a body oscillating with amplitude A at the angle α to the vertical. This output, given by

$$\langle P \rangle = \tfrac{1}{2}\rho_0 V \omega^3 A^2 \,\mathrm{Im}\left[C_\mathrm{h}^2 \sin^2\alpha + C_z^2 \cos^2\alpha\right],$$

is represented in figure 2(*a*) for the cylinder and sphere. For $\omega > N$, the waves are evanescent and no output is observed. For $\omega < N$, the waves are propagative and the output at given A is maximum at $\omega/N = \sqrt{(2/3)} \approx 0.82$ for the cylinder, and at ω/N varying weakly, between 0.84 and 0.85, with α for the sphere.

A second application is the oscillation $Z(t)$ of a body displaced slightly by Z_0 from its neutral buoyancy level then released at $t = 0$. The temporal Fourier transform of the oscillation is

$$\frac{Z(\omega)}{Z_0} = \frac{\mathrm{i}}{\omega}\frac{1 + C_z(\omega)}{1 + C_z(\omega) - N^2/\omega^2},$$

yielding $Z(t)/Z_0 = J_0(Nt)$ for the cylinder and $\frac{1}{2}\pi\mathbf{E}_1(Nt)$ for the sphere, with J_ν a Bessel function and $\mathbf{E}_\nu$ a Weber function. The oscillation, represented in figure 2(*b*), is consistent with direct calculation and measurement[18]. Experiments for larger initial displacements[19,20] have pointed out the importance of viscous damping, and the topic remains an area of active research[21–23].

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