

Added mass for wave motion in density-stratified fluids

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In a density-stratified fluid the existence of a restoring force – buoyancy – alters fundamentally the dynamics of submerged bodies. This effect can be described as a modification of their added mass. We investigate it in the simplest possible configuration – the small-amplitude motion of an unbounded Boussinesq uniformly stratified fluid – and for the simplest possible bodies – a horizontal circular cylinder in two dimensions and a sphere in three dimensions.

General case The concept of added mass pertains to the irrotational flow of a homogeneous fluid¹. It follows from the linearity of the flow, such that the translation of a rigid body at the velocity \mathbf{U} creates a velocity potential $\phi = \phi_i U_i$. By defining the added mass tensor m_{ij} according to

$$m_{ij} = \rho \oint_S n_i \phi_j \, d^2S,$$

with ρ the density of the fluid, S the surface of the body and \mathbf{n} the inward normal to S , we may express the pressure force on the body as

$$F_i = \oint_S p n_i \, d^2S = -\rho \frac{d}{dt} \oint_S \phi n_i \, d^2S = -m_{ij} \frac{dU_j}{dt},$$

the impulse and (kinetic) energy of the fluid as, respectively,

$$I_i = \rho \oint_S \phi n_i \, d^2S = m_{ij} U_j, \quad E = \frac{1}{2} \rho U_i \oint_S \phi n_i \, d^2S = \frac{1}{2} m_{ij} U_i U_j,$$

and the dipole strength of the body through

$$\rho \mathcal{D}_i = \rho \oint_S \left(n_i \phi - x_i \frac{\partial \phi}{\partial n} \right) \, d^2S = (m_f \delta_{ij} + m_{ij}) U_j,$$

with V the volume of the body and $m_f = \rho V$ the mass of the displaced fluid. Accordingly, added mass characterizes the flow fully, being involved in the dynamics of the body through \mathbf{F} , in the dynamics of the fluid as a whole through \mathbf{I} and E , and in the dynamics of the distant fluid through \mathcal{D} .

The small-amplitude Boussinesq motion of a uniformly stratified fluid of buoyancy frequency N can similarly be described^{2–4} in terms of a scalar function χ , satisfying the internal wave equation

$$\left(\frac{\partial^2}{\partial t^2} \nabla^2 + N^2 \nabla_h^2 \right) \chi = 0,$$

with the z -axis directed vertically upwards and the subscript $_h$ denoting a horizontal projection. The velocity \mathbf{u} and the disturbances p in pressure and ρ in density are related to the wave function through

$$\mathbf{u} = \left(\frac{\partial^2}{\partial t^2} \nabla + N^2 \nabla_h \right) \chi, \quad p = -\rho_0 \left(\frac{\partial^2}{\partial t^2} + N^2 \right) \frac{\partial \chi}{\partial t}, \quad \rho = \rho_0 \frac{N^2}{g} \frac{\partial^2 \chi}{\partial t \partial z},$$

with g the acceleration due to gravity, while the pressure p_0 and density ρ_0 at rest satisfy $dp_0/dz = -\rho_0 g$ and $d\rho_0/dz = -\rho_0 N^2/g$. As for irrotational flow, the linearity of the wave equation implies a linear relation between the velocity \mathbf{U} of a translating rigid body and the wave function that it creates, in the form of a temporal convolution $\chi = \chi_i * U_i$.

The pressure force on the body follows immediately as

$$F_i = \oint_S p n_i d^2S = -\rho_0 \frac{\partial}{\partial t} \left(\frac{\partial^2}{\partial t^2} + N^2 \right) \oint_S n_i \chi d^2S.$$

Equations of conservation may be derived for the momentum and total (kinetic and potential) energy of the fluid, of respective densities $\rho_0 u_i$ and $\frac{1}{2}\rho_0 u_i^2 + \frac{1}{2}\rho_0 N^2 \zeta^2$, with $\zeta = \partial^2 \chi / \partial t \partial z$ the vertical displacement of fluid particles. By integrating the associated fluxes over the surface of the body, we obtain the momentum and energy outputs of the body as, respectively,

$$I_i = \rho_0 \oint_S \left(n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi d^2S, \quad E = \rho_0 U_i \left(\frac{\partial^2}{\partial t^2} + N^2 \right) \oint_S n_i \chi d^2S.$$

Introduction of the Green's function of the wave equation⁵ combined with application of standard techniques^{6,7} yields a Kirchhoff–Helmholtz integral equation for internal waves, which corrects⁸ and generalizes⁹. By expanding this equation at large distances from the body, we obtain the dipole strength of the body, such that

$$\rho_0 \mathcal{D}_i = \rho_0 \oint_S \left[\left(n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi - x_i \left(\frac{\partial^2}{\partial t^2} \frac{\partial}{\partial n} + N^2 \frac{\partial}{\partial n_{hi}} \right) \chi \right] d^2S.$$

Accordingly, two distinct definitions of the added mass tensor may be proposed. One,

$$m_{ij}^{(1)} = \rho_0 \left(\frac{\partial^2}{\partial t^2} + N^2 \right) \oint_S n_i \chi_j d^2S,$$

involved in pressure and energy through

$$F_i = -m_{ij}^{(1)} * \frac{dU_j}{dt}, \quad E = U_i \left[m_{ij}^{(1)} * U_j \right],$$

and the other

$$m_{ij}^{(2)} = \rho_0 \oint_S \left(n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi_j d^2S,$$

involved in momentum and dipole strength through

$$I_i = m_{ij}^{(2)} * U_j, \quad \rho_0 \mathcal{D}_i = \left[m_{if} \delta_{ij} \delta(t) + m_{ij}^{(2)} \right] * U_j.$$

The relation between the two is simplified in the monochromatic case, when the excitation \mathbf{U} and the responses \mathbf{u} , p and ρ depend on time through the factor $e^{-i\omega t}$ which is suppressed in the following. The time-dependent added masses are replaced by their temporal Fourier transforms and the above relations assume the more familiar forms

$$F_i = i\omega m_{ij}^{(1)} U_j, \quad \langle E \rangle = \frac{1}{2} \text{Re} \left[\overline{U_i} m_{ij}^{(1)} U_j \right], \quad I_i = m_{ij}^{(2)} U_j, \quad \rho_0 \mathcal{D}_i = \left[m_{if} \delta_{ij} + m_{ij}^{(2)} \right] U_j,$$

where $\bar{}$ denotes a complex conjugate and $\langle \rangle$ a time average. The two added mass tensors are related anisotropically through

$$m_{ij}^{(1)} = \left(1 - \frac{N^2}{\omega^2} \delta_{i3} \right) m_{ij}^{(2)}.$$

As did^{10–12}, in the following we will consider the first definition only and omit the superscript⁽¹⁾.

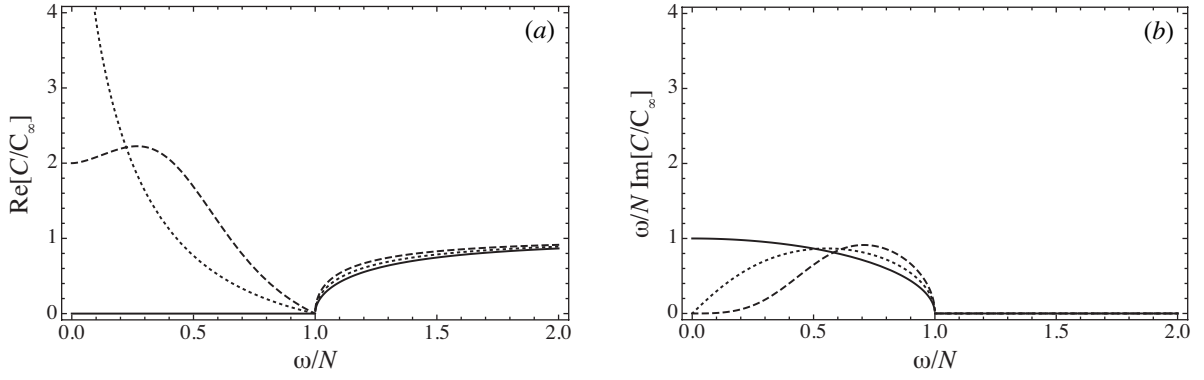


Figure 1 – Coefficients of (a) added mass and (b) damping for an oscillating circular cylinder (—) and a sphere oscillating horizontally (---) or vertically (.....).

Oscillating bodies We illustrate these results for two particular oscillating bodies, a horizontal circular cylinder and a sphere, represented as source terms $q = \sigma \delta_S$ on the right-hand side of the wave equation, namely as surface distributions of singularities of density σ . The condition of fixed normal velocity U_n on S becomes an integral equation for σ , solved by use of stretched orthogonal curvilinear coordinates¹³ and expansion into circular or spherical harmonics¹⁴. For rigid oscillation at the velocity \mathbf{U} the solution is found for the cylinder, of radius a , as

$$q(\mathbf{x}) = \left\{ \left[1 + \left(1 - \frac{N^2}{\omega^2} \right)^{1/2} \right] U_x \frac{x}{a} + \left[1 + \left(1 - \frac{N^2}{\omega^2} \right)^{-1/2} \right] U_z \frac{z}{a} \right\} \delta(r - a),$$

with $r = |\mathbf{x}|$, and for the sphere, also of radius a , as

$$q(\mathbf{x}) = \left[\frac{2}{1 + B(\omega/N)} \mathbf{U}_h \cdot \frac{\mathbf{x}_h}{a} + \frac{1}{1 - B(\omega/N)} U_z \frac{z}{a} \right] \delta(r - a),$$

with

$$B\left(\frac{\omega}{N}\right) = \frac{\omega^2}{N^2} \left[1 - \left(\frac{\omega^2}{N^2} - 1 \right)^{1/2} \arcsin\left(\frac{N}{\omega}\right) \right].$$

Complex square roots are fixed, in accordance with causality, by replacing ω by $\omega + i0$, in other words by adding to the real frequency ω a positive imaginary part which is later allowed to tend to zero.

The associated added mass coefficients $C_{ij} = m_{ij}/(\rho_0 V)$, such that $\mathcal{D}_i = \int x_i q(\mathbf{x}) d^3x = V[\delta_{ij} + C_{ij}/(1 - \delta_{i3} N^2/\omega^2)]U_j$, are complex. Their real part represents added inertia and their imaginary part, only present when $\omega < N$ namely when propagative waves are generated, represents wave damping. They are given for the cylinder by

$$C_h = C_z = \left(1 - \frac{N^2}{\omega^2} \right)^{1/2},$$

and for the sphere by

$$C_h = \frac{1 - B(\omega/N)}{1 + B(\omega/N)}, \quad C_z = \left(1 - \frac{N^2}{\omega^2} \right) \frac{B(\omega/N)}{1 - B(\omega/N)}.$$

Their variations, consistent with direct calculations^{10,12,15,16} and measurements¹⁷, are represented in figure 1. In the limit $\omega/N \rightarrow \infty$ the values $C_\infty = 1$ for the cylinder and $\frac{1}{2}$ for the sphere in a homogeneous fluid are recovered.

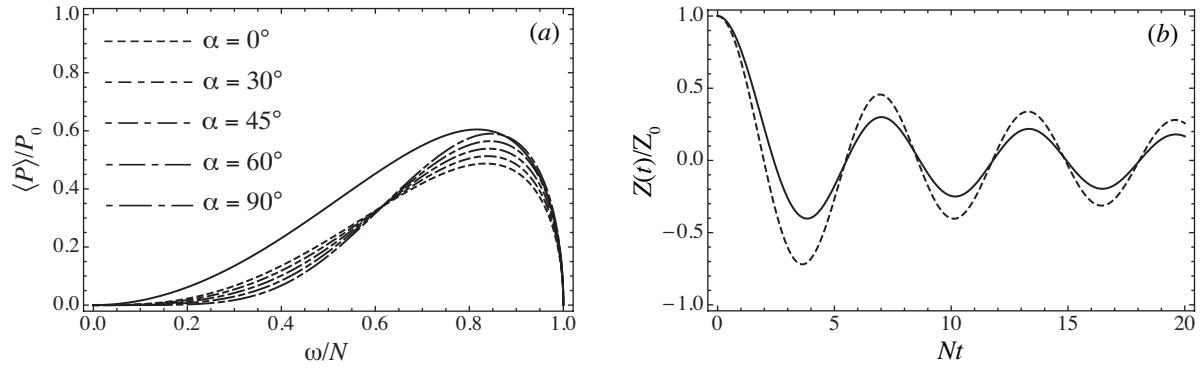


Figure 2 – For a circular cylinder (continuous line) and a sphere (dashed lines), (a) power output of forced oscillations, normalized by $P_0 = \rho_0 N^3 a^2 A^2$ for the cylinder and $\rho_0 N^3 a^3 A^2$ for the sphere, and (b) free buoyant oscillations.

Two applications A first application of the above is the average power output $\langle P \rangle$ of a body oscillating with amplitude A at the angle α to the vertical. This output, given by

$$\langle P \rangle = \frac{1}{2} \rho_0 V \omega^3 A^2 \text{Im} [C_h^2 \sin^2 \alpha + C_z^2 \cos^2 \alpha],$$

is represented in figure 2(a) for the cylinder and sphere. For $\omega > N$, the waves are evanescent and no output is observed. For $\omega < N$, the waves are propagative and the output at given A is maximum at $\omega/N = \sqrt{2/3} \approx 0.82$ for the cylinder, and at ω/N varying weakly, between 0.84 and 0.85, with α for the sphere.

A second application is the oscillation $Z(t)$ of a body displaced slightly by Z_0 from its neutral buoyancy level then released at $t = 0$. The temporal Fourier transform of the oscillation is

$$\frac{Z(\omega)}{Z_0} = \frac{i}{\omega} \frac{1 + C_z(\omega)}{1 + C_z(\omega) - N^2/\omega^2},$$

yielding $Z(t)/Z_0 = J_0(Nt)$ for the cylinder and $\frac{1}{2}\pi \mathbf{E}_1(Nt)$ for the sphere, with J_ν a Bessel function and \mathbf{E}_ν a Weber function. The oscillation, represented in figure 2(b), is consistent with direct calculation and measurement¹⁸. Experiments for larger initial displacements^{19,20} have pointed out the importance of viscous damping, and the topic remains an area of active research^{21–23}.

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