A Time-domain Model of Internal Wave Diffraction from a 3D Body in a Two-layer Fluid

B Teng, Y Gou

(State Key Laboratory of Costal and Offshore Engineering, Dalian University of Technology, Dalian, China)

e-mail: bteng@dlut.edu.cn, gouying@dlut.edu.cn

INTRODUCTION

Density of sea water is actually changing with the depth due to the variations in salinity and temperature in the water depth direction. In the deep water area, the change in water density is evident, and internal waves may be generated inside stratification water. This phenomenon may induce potential risk to deep water engineering structures.

The simplest model for internal waves is the two-layer fluid model. In this model there exists a density discontinuity at the interface between the upper and the lower layers, and the density is constant in each layer. Ten and Kashiwagi (2006) used boundary integral-equation method, and developed a linearized 2-D radiation model. Yeung and Nguyen (1999) derived the Green functions in two-layer fluid of finite depth for 3-D problems.

All of those studies are carried out in the frequency domain. In this paper, a time-domain model is developed for internal wave diffraction from a 3-D body located in the upper layer fluid. The method uses simple Green functions, and is implemented with higher-order boundary element method (Teng, et al, 2006). Comparisons are made with an analytic solution for a truncated cylinder, and examination shows that the model gives very steady results and has good agreement with the analytic solution.

NUMERICAL MODEL

A Cartesian coordinate system is defined with the origin in the plane of the undisturbed free surface, and the z-axis positive upwards. Other notations are shown in Fig.1. The densities of the fluids in the upper and the lower layers are ρ_1 and ρ_2, respectively. The fluid in each layer is assumed to be inviscid and incompressible, and the flow irrotational. So the velocity potential $\Phi^1(x,y,z,t)$ and $\Phi^2(x,y,z,t)$ in the fluid domain Ω_1 and Ω_2 satisfy Laplace equation.
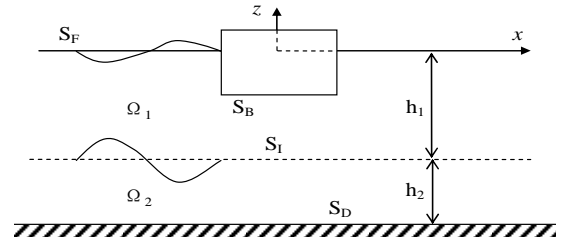

Fig.1 Definition Sketch

Under the assumption of small wave steepness, the linearized boundary conditions are satisfied as follows:

$$\phi^1_{tt} + g\phi^1_z = 0, \qquad \text{on z=0} \tag{1}$$

$$\phi^1_z = \phi^2_z, \qquad \text{on z=-h}_1 \tag{2}$$

$$\gamma(\phi^1_{tt} + g\phi^1_z) = \phi^2_{tt} + g\phi^2_z, \qquad \text{on z=-h}_1 \tag{3}$$

$$\phi^2_z = 0, \qquad \text{on z=-h}_1\text{-h}_2 \tag{4}$$

where $\gamma = \rho_1/\rho_2$.

The wave elevation on the free surface η^1 and on the interface η^2, have the relations as following:

$$\phi^1_t = -g\eta^1, \qquad \text{on z=0} \tag{5}$$

$$\eta^2_t = \phi^2_z, \qquad \text{on z=-h}_1 \tag{6}$$

For simplicity, we divide the potentials into the incident and the diffraction potentials in the form:

$$\Phi^1 = \phi^1_i + \phi^1_d, \quad \Phi^2 = \phi^2_i + \phi^2_d \tag{7}$$

The incident potentials and the wave elevations are as follows:

$$\phi_i^1 = \frac{\omega A\left(\dfrac{k_0}{k}\sinh kz + \cosh kz\right)}{k_0 \cosh kh_1 - k\sinh kh_1}\sin(kx - \omega t) \tag{8}$$

$$\phi_i^2 = \frac{\omega A \cosh k(z+h)}{k\sinh kh_2}\sin(kx - \omega t) \tag{9}$$

$$\eta_i^1 = A_1 \cos\left(k(x\cos\beta + y\sin\beta) - \omega t\right) \tag{10}$$

$$\eta_i^2 = A\cos\left(k(x\cos\beta + y\sin\beta) - \omega t\right) \tag{11}$$

$$\frac{\eta_i^1}{\eta_i^2} = \cosh kh_1\left[1 - \frac{k}{k_0}\tanh kh_1\right] \tag{12}$$

where $A_1 = A/\cosh kh_1\left[1 - k/k_0 \tanh kh_1\right]$,

$k_0 = \omega^2/g$, k and k_0 satisfy the dispersion relations as follows:

$$\begin{aligned}&k_0(k\sinh kh - k_0\cosh kh)\\&+(1-\gamma)\sinh kh_1\sinh kh_2(k_0^2 - k^2) = 0\end{aligned} \tag{13}$$

The above equation has two roots, and they are corresponding to the 'surface wave mode' and the 'internal wave mode', respectively. Eq. (13) can be simplified as follows when h_1 and h_2 are great enough:

$$(\frac{k}{k_0})^2(1-\gamma) - 2\frac{k}{k_0} + (\gamma+1) = 0 \tag{14}$$

The two roots of Eq. (14) are $\dfrac{k}{k_0} = 1$, $\dfrac{k}{k_0} = \dfrac{(1+\gamma)}{(1-\gamma)}$.

It can be seen obviously that the first root corresponding to the surface wave mode, and the other corresponding to the internal wave mode. In this paper we only concern to the internal wave mode.

For the diffraction potential, the linerazed boundary conditions are satisfied as follows:

$$\phi_{dz}^1 = -\phi_{iz}^1 \quad \text{on } S_B \tag{15}$$

$$\phi_{dz}^2 = 0 \quad \text{on } z=-h_1-h_2 \tag{16}$$

$$\begin{cases}\eta_{dt}^1 = \phi_{dz}^1 & \text{(a)}\\ \phi_{dt}^1 = -g\eta_d^1 & \text{(b)}\end{cases} \quad \text{on } z = 0 \tag{17}$$

$$\begin{cases}\eta_{dt}^2 = \phi_{dz}^2 & \text{(c)}\\ \phi_{dz}^1 = \phi_{dz}^2 & \text{(d)} \quad \text{on } z = -h_1 \\ \gamma(\phi_{dt}^1 + g\eta_d^2) = \phi_{dt}^2 + g\eta_d^2 & \text{(e)}\end{cases} \tag{18}$$

We define $\varphi = \gamma\phi_d^1 - \phi_d^2$, then the Eq.(18) can be rewritten as:

$$\varphi_t = (1-\gamma)g\eta_d^2 \quad \text{on } z = -h_1 \tag{19}$$

We assume the body is located in the upper fluid. Applying the Green's second identity to Green function and diffraction velocity potential ϕ_d^1 and ϕ_d^2 in each layer respectively, we can obtain the integral equations:

$$\begin{aligned}&\alpha\phi_d^1 - \iint_{S_B}\phi_d^1\frac{\partial G_1}{\partial n}ds + \iint_{S_F}G_1\frac{\partial\phi_d^1}{\partial n}ds + \iint_{S_I}G_1\frac{\partial\phi_d^1}{\partial n}ds\\ &= -\iint_{S_B}G_1\frac{\partial\phi_d^1}{\partial n}ds + \iint_{S_F}\phi_d^1\frac{\partial G_1}{\partial n}ds + \iint_{S_I}\phi_d^1\frac{\partial G_1}{\partial n}ds\end{aligned} \tag{20}$$

and

$$\iint_{S_I}G_2\frac{\partial\phi_d^2}{\partial n}ds = \iint_{S_I}\phi_d^2\frac{\partial G_2}{\partial n}ds - \alpha\phi_d^2 \tag{21}$$

where

$$G_1 = -\frac{1}{4\pi r}, \quad G_2 = -\frac{1}{4\pi r} - \frac{1}{4\pi r_2} \tag{22}$$

r is the distance between the field and the source points, and r_2 is the distance between the field point and the image of the source point about the sea bed.

The integral equation in the upper fluid includes the integration over the body surface, the free surface and the interface between the two layer of fluids. However, the integral equation in the lower fluid only includes the integration over the interface between the two layer of fluids.

After discretization of Eqs. (20) and (21), we obtain two sets of linear equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \left\{ \dfrac{\partial \phi_d^1}{\partial n} \right\}_{S_F} \\ \left\{ \phi_d^1 \right\}_{S_B} \\ \left\{ \dfrac{\partial \phi_d^1}{\partial n} \right\}_{S_I} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{Bmatrix} \left\{ \phi_d^1 \right\}_{S_F} \\ \left\{ \dfrac{\partial \phi_d^1}{\partial n} \right\}_{S_B} \\ \left\{ \phi_d^1 \right\}_{S_I} \end{Bmatrix} \quad (23)$$

$$[b] \left\{ \frac{\partial \phi_d^2}{\partial n} \right\}_{S_I} = [t] \left\{ \phi_d^2 \right\}_{S_I} \quad (24)$$

Applying the interface conditions Eqs.(18) and (19), we can combine Eqs. (23) and (24) to get a single set of linear equations as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} + \dfrac{1}{\gamma} s_{13} t^{-1} b \\ a_{21} & a_{22} & a_{23} + \dfrac{1}{\gamma} s_{23} t^{-1} b \\ a_{31} & a_{32} & a_{33} + \dfrac{1}{\gamma} s_{33} t^{-1} b \end{bmatrix} \begin{Bmatrix} \left\{ \dfrac{\partial \phi_d^1}{\partial n} \right\}_{S_F} \\ \left\{ \phi_d^1 \right\}_{S_B} \\ \left\{ \dfrac{\partial \phi_d^1}{\partial n} \right\}_{S_I} \end{Bmatrix} =$$

$$\begin{bmatrix} s_{11} & s_{12} & \dfrac{1}{\gamma} s_{13} \\ s_{21} & s_{22} & \dfrac{1}{\gamma} s_{23} \\ s_{31} & s_{32} & \dfrac{1}{\gamma} s_{33} \end{bmatrix} \begin{Bmatrix} \left\{ \phi_d^1 \right\}_{S_F} \\ \left\{ \dfrac{\partial \phi_d^1}{\partial n} \right\}_{S_B} \\ \left\{ \varphi \right\}_{S_I} \end{Bmatrix} \quad (25)$$

In the time matching procedure, the 4th-order Runga-Kutta approach is used, basing on boundary conditions Eqs.(17), (18) and (19). Then, the time history of velocity potential on the body surface can be obtained. With the integration of wave pressure over the body surface, the internal wave force and moment can be obtained.

NUMERICAL RESULTS

A truncated cylinder in a two layer fluid has been calculated using the proposed method. The sketch is shown in Fig.1. The water depths of the upper and the lower layers are $h_1/h=0.7$, $h_2/h=0.3$. The densities of the fluids in the upper and the lower layers are $\rho_1 =998.2 \text{kg/m}^3$ and $\rho_2 =1027.2 \text{kg/m}^3$. That means $\gamma =0.97$. The cylinder has a radius of $a/h=0.5$, and a draft of $T/h=0.5$. Figs.2-4 show the wave force and the

wave moment on the cylinder at the wave number kh=3.5. The dimensionless factor for wave forces is $\rho_1 gahA_1$, and for moment is $\rho_1 gah^2 A_1$. It can be seen that the results are very steady.
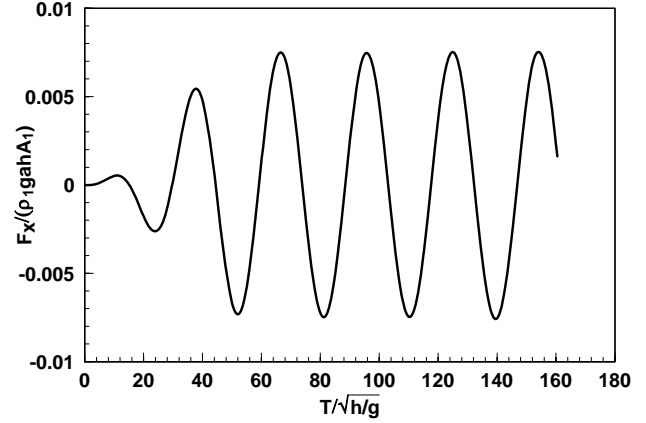

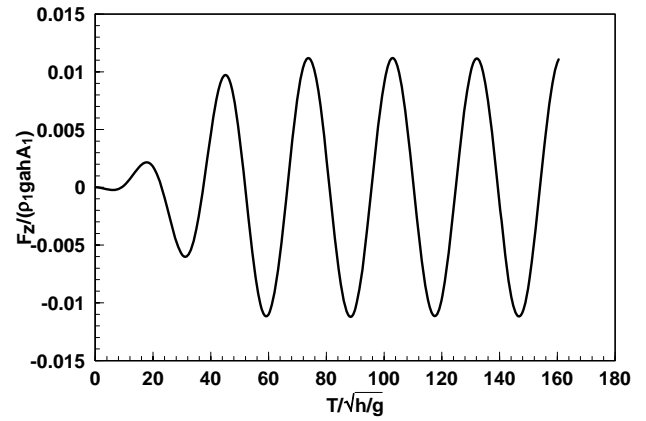Fig.2 Time histories of horizontal wave force

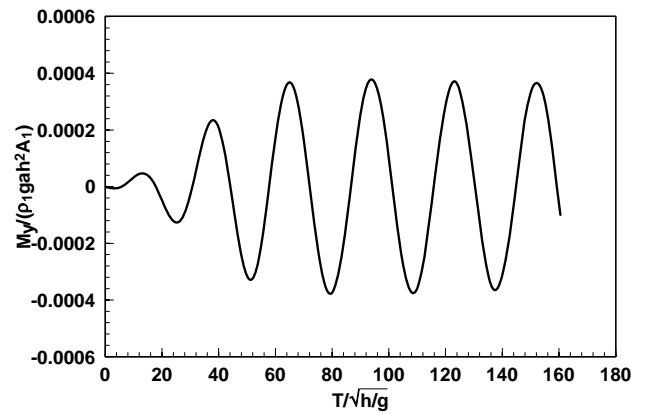Fig.3 Vertical wave force time histories

Fig.4 Wave moment time histories

To check the accuracy, the same cylinder has been calculated. And the results are compared with the analytical results of You, et al (2007). From Figs.5-7 we can see that the present results have good agreement with the analytical solutions.
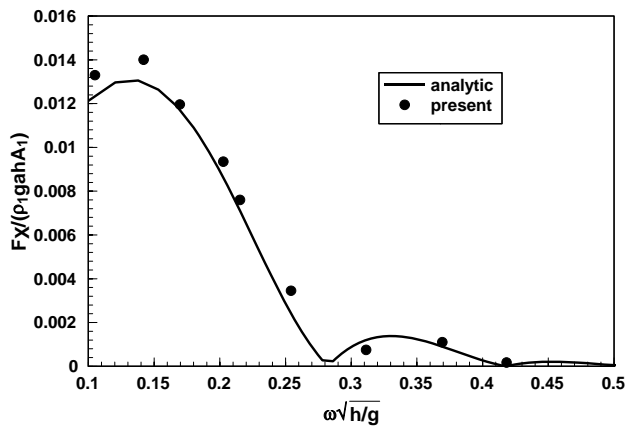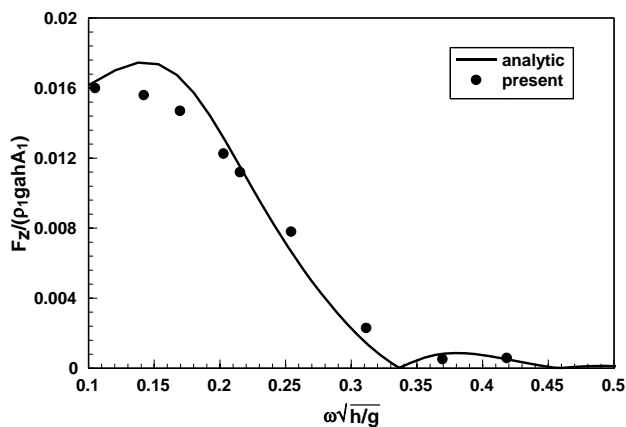

Fig.5 Horizontal wave force on cylinder

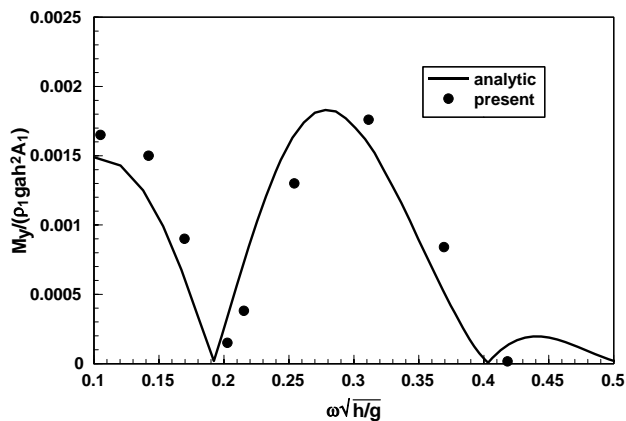Fig.6 Vertical wave force on cylinder

Fig.7 Vertical wave force on cylinder

CONCLUSIONS

In this paper, a 3D time-domain model for internal wave diffraction in a two-layer fluid is developed. Through combining the two integral equations in the upper and the lower layers, a single set of linear equations are set up to compute the time histories of internal wave potential and wave profiles. From the results of internal wave force and moment on a truncated cylinder, it can be seen that this method have good agreement with analytical results.

References

Igor Ten, Masashi Kashiwagi.(2004) Hydrodynamics of a body floating in a two-layer fluid of finite depth. Part 1. radiation problem. Journal of Marine Science and Technology 9:127-141

Yeung RW, Nguyen T (1999) Radiation and diffraction of waves in a two-layer fluid. In: Proceedings of the 22nd Symposium on Naval Hydrodynamics, Washington, D.C., pp 875-891

You YX, Shi Q, Wei G, Miao GP (2007) The interaction of water waves with a vertically floating cylinder in a two-layer fluid. Journal of Shanghai Jiao Tong University, 41(9):1460-1464

Teng B, Gou Y and Ning DZ (2006) A higher order BEM for wave-current action on structures --- Direct computation of free-term coefficient and CPV integrals. China Ocean Engineering, 20(3):395-410