NONLINEAR HYDROELASTICITY OF A PLATE FLOATING ON SHALLOW WATER OF VARIABLE DEPTH

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1. Introduction

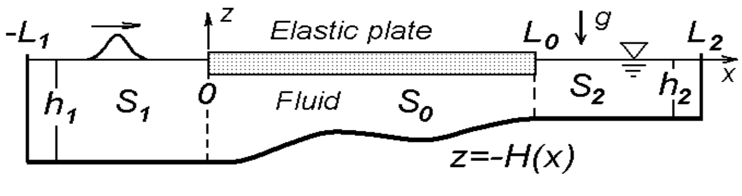
Although there is an extensive literature focused upon hydroelastic analysis of the floating platform, relatively little work has been done on dynamic response of a floating structure when subjected to nonlinear waves, as pointed out in [1]. The time domain analysis of the hydroelastic deformation of a floating rectangular plate was been fulfilled in [2]. A finite element method (FEM) was developed for 3D free-surface flows within the potential theory with nonlinear free-surface conditions. Similar 2D problem was solved in [3], where the boundary element method was applied to fluid motion and FEM to analysis of elastic deformations of a structure.

The unsteady hydroelastic problem is simplified for a plate floating on shallow water. The interaction between a solitary wave and a 2D floating elastic plate was studied in [4]. The matched asymptotic expansion method was used to connect the outer solution governed by the Boussinesq equations and the inner solution governed by the Laplace equation. The sets of equations were solved by the finite-difference method (FDM). Recently, the 2D nonlinear model based on the Level I Green-Naghdi equations was proposed in [5]. The resulting governing equations, subjected to the boundary and jump conditions, are solved by FDM. In all of these studies a flat seabed was assumed.

The aim of this paper is to consider the 2D unsteady hydroelasticity problem for a plate floating on shallow water of variable depth. Spectral method is used to solve this problem within the classical nonlinear shallow water theory (Airy's theory) and Boussinesq's theory. Proposed method can be used for any unsteady 2D problem, but here the scattering of solitary wave by an elastic plate is studied. Time-dependent hydroelastic response of a plate floating on shallow water of variable depth within the linear wave theory was investigated earlier in [6].

2. Mathematical formulation

An elastic thin plate floats on the surface of an inviscid incompressible fluid layer in the tank with the vertical side boundaries. The plate is infinite in the y-direction, so that only the x- and z-directions are considered. The x-direction is horizontal, the positive z-axis points vertically up, and the plate covers the region $0 \le x \le L_0$. It is assumed that there is not air gap between the plate bottom and fluid. The surface of the fluid that is not covered with the plate is free. The whole domain of the fluid, $-L_1 \le x \le L_2$, is divided into three parts: S_0 $(0 \le x \le L_0)$, S_1 $(-L_1 \le x < 0)$, S_2 $(L_0 < x \le L_2)$. Without the plate, the fluid depth is equal to $H(x)$ in S_0. For simplicity we assume that the sea floor is flat in the left- and right-hand domains S_1 and S_2, and the fluid depths in these domains are equal to h_1 and h_2, respectively. With the plate, the fluid depth in domain S_0 is equal to $h_0(x) = H(x) - d$, where d is the draft of the plate. The coordinate and fluid-structure systems of the 2D problem is shown on Figure.



It is assumed that the maximal depth of the fluid is small compared to the horizontal length of the domain and the length of surface waves, which makes it possible to use the shallow-water wave theory. The vertical deflection of the elastic plate is assumed to be governed by the linear-plate theory, because a wavelength of flexural-gravity waves is greater than the wavelength of surface waves. As noted in [7], nonlinearity in the hydrodynamics dominates nonlinearity arising in the plate.

The governing equations for the motion of the fluid can be written as follows

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(h+\eta)u] = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \varepsilon\left[\frac{h}{2}\frac{\partial^2}{\partial x^2}\left(h\frac{\partial u}{\partial t}\right) - \frac{h^2}{6}\frac{\partial^3 u}{\partial t \partial x^2}\right], \tag{2}$$

where $\varepsilon = 1$ corresponds to the Boussinesq's theory and $\varepsilon = 0$ to the Airy's theory (see, for example, [8]), ρ is the fluid density, g is the gravity acceleration, $\eta(x,t)$ is the fluid surface deviation, $u(x,t)$ is the horizontal velocity of fluid particles in the x-direction, $h(x)$ is the still fluid depth, $h(x) = h_0(x)$ in the domain S_0 and $h = h_1$ and $h = h_2$ in domains S_1 and S_2, respectively, $p(x,t)$ is the pressure on the upper surface of water, p is zero in the domains S_1 and S_2, and in domain S_0

$$p = m\frac{\partial^2 \eta}{\partial t^2} + D\frac{\partial^4 \eta}{\partial x^4} + mg, \tag{3}$$

where $m = \rho d$ is the mass per unit length of the plate, D is the flexural rigidity of the plate. Using the mass continuity, Eq. (1), it is possible to reduce the order of time derivative in Eq. (3)

$$\frac{\partial^2 \eta}{\partial t^2} = -\left[\frac{\partial u}{\partial t}\frac{\partial (h+\eta)}{\partial x} + u\frac{\partial^2 \eta}{\partial t \partial x} + \frac{\partial \eta}{\partial t}\frac{\partial u}{\partial x} + (h+\eta)\frac{\partial^2 u}{\partial t \partial x}\right]. \tag{4}$$

The vertical side walls $x = -L_1$ and $x = L_2$ are the open boundaries, at which the wave energy is dissipated. The open boundary condition are

$$\frac{\partial \Omega}{\partial t} - \sqrt{gh_1}\frac{\partial \Omega}{\partial x} = 0 \quad (x = -L_1), \qquad \frac{\partial \Omega}{\partial t} + \sqrt{gh_2}\frac{\partial \Omega}{\partial x} = 0 \quad (x = L_2), \tag{5}$$

where Ω is η or u.

At the plate edges, free edge boundary conditions require vanishing bending moment and shear force: $\partial^2\eta/\partial x^2 = \partial^3\eta/\partial x^3 = 0 \quad (x = 0,\ x = L_0)$.

The jump conditions are needed to match the solutions at the interfaces between the domains S_0 and S_1, $x = x_l^{\pm}$, as well as S_0 and S_2, $x = x_r^{\pm}$, where $x_l = 0$ and $x_r = L_0$. This is because the floating structure causes discontinuities of the fluid layer thickness, depth averaged velocity and integrated pressure over depth at the cross sections between open water and the edges of the plate. Based on the conservation laws of mass, horizontal momentum, moment of vertical momentum and energy conservation, the jump conditions are (see for more details [5])

$$[\![(h+\eta)u]\!]_{x_l} = 0, \quad [\![(h+\eta)u]\!]_{x_r} = 0, \tag{6}$$

$$[\![V]\!]_{x_l} = \frac{\rho}{3}\frac{\partial[u(h+\eta)]}{\partial x}\bigg|_{x_l^+}[\![v]\!]_{x_l}, \qquad [\![V]\!]_{x_r} = -\frac{\rho}{3}\frac{\partial[u(h+\eta)]}{\partial x}\bigg|_{x_r^-}[\![v]\!]_{x_r}, \tag{7}$$

$$V = \frac{1}{2}\rho[u^2 + \frac{1}{3}v^2 + g(h+\eta)] + \frac{P}{h+\eta}, \quad v = (h+\eta)\frac{\partial u}{\partial x}.$$

Here P is the integrated pressure through the water column

$$P = (h+\eta)\left\{\frac{g\rho}{2}(h+\eta) + \varepsilon\frac{\rho}{6}[(h+\eta)^2 - 3h^2]\frac{\partial^2 u}{\partial t \partial x} + p\right\},$$

and the notation is used $[\![f]\!]_x = f|_{x^+} - f|_{x^-}$.

It is assumed that at the initial time the plate and the fluid in domains S_0 and S_2 are at rest. In domain S_1, a solitary wave is assumed traveling to the right. Initial conditions have the form as in [5]

$$\eta(x,0) = \frac{\alpha}{\cosh^2 A}, \quad u(x,0) = \frac{\sqrt{g(h_1+\alpha)}}{h_1+\eta}\eta \quad (x \in S_1), \quad A = \sqrt{\frac{3\alpha}{4(h_1+\alpha)}}\frac{x-x_0}{h_1}, \tag{8}$$

$$\eta(x,0) = u(x,0) = 0 \quad (x \notin S_1),$$

where α is the wave height above still water level, and x_0 is the initial location of the wave peak.

Non-dimensional variables are introduced and used below with L_0 as length scale, and $\sqrt{L_0/g}$ as time scale.

3. Spectral method

By virtue of the fact that there are discontinuities of the fluid layer thickness and depth averaged velocity at $x = 0$ and $x = 1$, the next designations are used:

$$H_0(x,t) = h_0(x) + \eta(x,t), \quad u_0(x,t) = u(x,t) \quad (x \in S_0),$$

$$H_j(x,t) = h_j + \eta(x,t), \quad u_j(x,t) = u(x,t) \quad (x \in S_j), \quad (j = 1,2).$$

The plate deflection is sought in the form of an expansion in the eigenfunctions of vibrations of a free-edge beam in vacuum

$$H_0(x,t) = h_0(x) + \sum_{n=1}^{N} a_n(t) W_n(x). \tag{9}$$

Here the functions $a_n(t)$ are to be determined and the functions $W_n(x)$ are solutions of the spectral problem in non-dimensional variables:

$$W_n^{(IV)} = \lambda_n^4 W_n \quad (0 \le x \le 1), \quad W_n'' = W_n''' = 0 \quad (x = 0,\ x = 1).$$

The prime denotes differentiation with respect to x. The functions $W_n(x)$ are well known (see, for example, [6]).

Other unknown functions are sought in the forms of truncated Fourier series:

$$u_0(x,t) = \frac{q_2(t)}{H_0(0,t)}(1-x) + \frac{x q_3(t)}{H_0(1,t)} + \sum_{k=1}^{K} b_k(t) \sin k\pi x, \tag{10}$$

$$H_1(x,t) = \frac{1}{2} c_0(t) + \sum_{m=1}^{M} c_m(t) \cos \frac{m\pi x}{L_1}, \tag{11}$$

$$u_1(x,t) = \frac{q_2(t)}{H_1(0,t)}\left(1 + \frac{x}{L_1}\right) - \frac{x q_1(t)}{L_1 H_1(-L_1,t)} + \sum_{m=1}^{M} f_m(t) \sin \frac{m\pi x}{L_1}, \tag{12}$$

$$H_2(x,t) = \frac{1}{2} r_0(t) + \sum_{j=1}^{J} r_j(t) \cos \frac{j\pi(L_2 - x)}{L_3}, \tag{13}$$

$$u_2(x,t) = \frac{(x-1) q_4(t)}{L_3 H_2(L_2,t)} + \frac{(L_2 - x) q_3(t)}{L_3 H_2(1,t)} + \sum_{j=1}^{J} s_j(t) \sin \frac{j\pi(L_2 - x)}{L_3}, \tag{14}$$

where $L_3 = L_2 - 1$, and the unknown functions $q_l(t)$ $(l = 1 \div 4)$ correspond to the water discharges through the sections $x = -L_1,\ x = 0,\ x = 1,\ x = L_2$, respectively. The total number of unknown functions is equal to $N_s = N + K + 2(M + J) + 6$.

We substitute the expansions (9) and (10) into Eq. (1), multiply the obtained relation by $W_l(x)$ $(l = 1, \dots, N)$, and integrate its over x from 0 to 1. Them we substitute the expansions (9) and (10) into Eq. (2) taking into account Eqs. (3) and (4), multiply the obtained relation by $\sin l\pi x$ $(l = 1, \dots, K)$, and integrate its over x from 0 to 1.

In a similar way for the domain S_1, we substitute the expansions (11) and (12) into Eqs. (1) and (2), multiply the obtained relations by $\cos(l\pi x/L_1)$ $(l = 0, \dots, M)$ for Eq. (1) and by $\sin(l\pi x/L_1)$ $(l = 1, \dots, M)$ for Eq. (2), and integrate them over x from $-L_1$ to 0. For the domain S_2, we substitute the expansions (13) and (14) into Eqs. (1) and (2), multiply the obtained relations by $\cos[l\pi(L_2 - x)/L_3]$ $(l = 0, \dots, J)$ for Eq. (1) and by $\sin[l\pi(L_2 - x)/L_3]$ $(l = 1, \dots, J)$ for Eq. (2), and integrate them over x from 1 to L_2.

As a result, we obtain the set of $N_s - 4$ ordinary differential equations (ODE's) of the first order. This set is to be supplemented with four differential equations which come from the two boundary conditions (5) and two jump conditions (7) taking into consideration Eqs. (3) and (4). The jump conditions (6) are satisfied automatically thanks to special forms of expansions (10), (12) and (14).

The final set of ODE's is written in the matrix form

$$\mathbf{C}\dot{\mathbf{Y}} = \mathbf{B}, \tag{15}$$

where

$$\mathbf{Y} = \{a_1, \ldots, a_N; b_1, \ldots, b_K; c_0, c_1, \ldots, c_M; f_1, \ldots, f_M; r_0, r_1, \ldots, r_J; s_1, \ldots, s_J; q_1, \ldots, q_4\}^T,$$

an overdot denotes differentiation with respect to time, and the superscript T denotes the transposition. The quadratic matrix $\mathbf{C}$ and the vector $\mathbf{B}$ are nonlinear functions of the vector $\mathbf{Y}$.

The initial conditions for Eq. (15) can be written, in view of Eqs. (8), (11) and (12), as

$$c_m(0) = \frac{2}{L_1} \int_{-L_1}^{0} H_1(x, 0) \cos \frac{m\pi x}{L_1} \mathrm{d}x \quad (m = 0, 1, \ldots, M),$$

$$f_m(0) = \frac{2}{L_1} \int_{-L_1}^{0} u_1(x, 0) \sin \frac{m\pi x}{L_1} \mathrm{d}x \quad (m = 1, \ldots, M).$$

All other components of the vector $\mathbf{Y}$ are equal to zero at $t = 0$.

For the numerical solution, the set of ODE's (15) is rewrote as $\dot{\mathbf{Y}} = \mathbf{C}^{-1}\mathbf{B}$ and is solved using the 4-th order Runge - Kutta scheme.

The plate deflections and wave motion of the fluid for various bottom topographies and the amplitudes of the incident solitary wave will be presented at the Workshop. The linear and nonlinear responses of the elastic plate and the fluid will be compared.

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