Water-wave scattering by vast fields of bodies such as ice floes in the Marginal Ice Zone

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1 Introduction

The Marginal Ice Zone (MIZ) is an interfacial region which forms at the boundary of open and frozen ocean. It consists of vast fields of ice floes scattering the incoming ocean waves and it is of great importance to climate research to understand wave propagation and scattering and wave-induced break up in the MIZ (see Squire, 2007, for more information).

We present here a computationally efficient method to calculate the scattering of vast fields of bodies within the framework of linear theory. It is the aim to apply this method to approximate the scattering characteristics of the MIZ without having to assume the problem is two-dimensional but the method is general and applies to a variety of situations. The idea is briefly summarised as follows: We group several bodies into modules and determine the scattering characteristics of an infinite periodic line array of such modules. The field of bodies is then assembled by placing many infinite line arrays behind one another. A sketch of the geometry is given in figure 1. Ultimately we aim to use this method to determine the attenuation coefficient which we can compare to existing two-dimensional theory (Kohout & Meylan, 2008) or to experimental measurements.

It was described in Meylan (2002) how to calculate the scattering by a single ice floe of arbitrary geometry modelled as a thin elastic plate. An interaction theory (Kagemoto & Yue, 1986; Peter & Meylan, 2004) provides an efficient exact algebraic method for calculating the scattering by finitely many bodies. Grouping such an arrangement of bodies into a module, the scattering by an infinite periodic line array of modules is easily calculated using the method described in Peter *et al.* (2006). There, it was shown that the scattered wave field away from the array consists of plane waves propagating in a finite number of directions and that this far-field approximation is accurate even near the array. Using an idea of McPhedran *et al.* (1999), developed for electromagnetic scattering by cylinders, we present an explicit iterative method allowing the scattering characteristics of a stack of many arrays placed one behind the other to be calculated. This method requires that the arrays have the same periodic spacing, but does not require that the arrays are identical nor that the spacing between arrays is constant. This allows us to perform averaging, which is important to remove resonance effects.

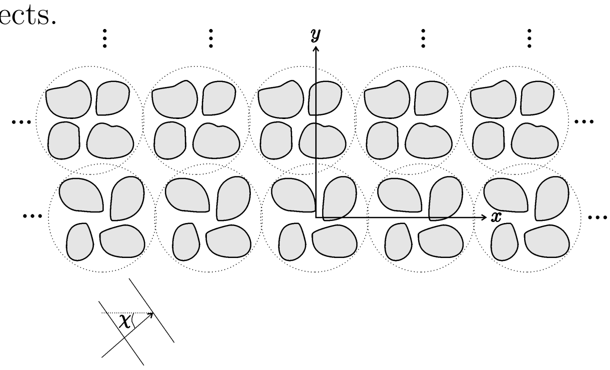


Fig. 1: Sketch of the geometry (plan view). Many periodic arrays of modules form the stack approximating part of the MIZ.

2 Statement of the problem

We consider the water-wave scattering of a plane wave by vertically non-overlapping bodies. The ambient plane wave is assumed to travel in the direction $\chi \in (0, \pi)$, where χ is measured with respect to the x-axis. Let (r_j, θ_j, z) be the local cylindrical coordinates of the jth body, Δ_j. The global coordinates, centred at the origin, are denoted by (x, y, z) (Cartesian) or (r, θ, z) (cylindrical).

The equations of motion for the water are derived from the linearised inviscid theory assuming irrotational motion. Restricting to time-harmonic motion with radian frequency ω, the velocity potential Φ can be expressed as the real part of a complex quantity, $\Phi(\mathbf{y}, t) =$

$\mathrm{Re}\{\phi(\mathbf{y})e^{-i\omega t}\}$. To simplify notation, $\mathbf{y} = (x, y, z)$ always denotes a point in the water, which is assumed to be of constant finite depth d, while $\mathbf{x}$ always denotes a point of the undisturbed water surface assumed at $z = 0$.

Writing $\alpha = \omega^2/g$, where g is the acceleration due to gravity, the potential ϕ has to satisfy the standard boundary-value problem

$$\nabla^2\phi = 0, \qquad \mathbf{y} \in D, \tag{1a}$$
$$\frac{\partial\phi}{\partial z} = \alpha\phi, \qquad \mathbf{x} \in \Gamma^{\mathrm{f}}, \tag{1b}$$
$$\frac{\partial\phi}{\partial z} = 0, \qquad \mathbf{y} \in D,\ z = -d, \tag{1c}$$

where $D = (\mathbb{R}^2 \times (-d, 0)) \setminus \bigcup_j \overline{\Delta}_j$ is the domain occupied by the water and Γ^{f} is the free water surface. At the immersed body surface, the water velocity potential has to equal the normal velocity of the body. For an ice floe at the water surface, modelled as a thin elastic plate as in Meylan (2002), this reads

$$D\nabla^4 w - \omega^2\rho_\Delta h w = \mathrm{i}\omega\rho\phi - \rho g w, \quad \mathbf{x} \in \Delta_j, \tag{1d}$$

with complex floe displacement $w(\mathbf{x})$, water density ρ, ice rigidity D, density ρ_Δ and thickness h. Moreover, a radiation condition is imposed ensuring that there are only outgoing waves from each scatterer and we denote the ambient incident potential by $\phi^{\mathrm{In}} = \frac{Ag}{\omega} f_0(z) e^{\mathrm{i}k(x\cos\chi + y\sin\chi)}$. The positive wavenumber k is related to α by the dispersion relation $\alpha = k\tanh kd$, and the values of k_m, $m > 0$, are given as positive real roots of the dispersion relation $\alpha + k_m \tan k_m d = 0$. For ease of notation, we write $k_0 = -\mathrm{i}k$.

2.1 Eigenfunction expansion

The scattered potential of a body Δ_j can be expanded in singular cylindrical eigenfunctions,

$$\phi_j^{\mathrm{S}} = \sum_{m=0}^{\infty} f_m(z) \sum_{\mu=-\infty}^{\infty} A_{m\mu}^j K_\mu(k_m r_j) e^{\mathrm{i}\mu\theta_j}, \tag{2}$$

with discrete coefficients $A_{m\mu}^j$, where $f_m(z) = \frac{\cos k_m(z+d)}{\cos k_m d}$. The incident potential upon body Δ_j can be also be expanded in regular cylindrical eigenfunctions,

$$\phi_j^{\mathrm{I}} = \sum_{n=0}^{\infty} f_n(z) \sum_{\nu=-\infty}^{\infty} D_{n\nu}^j I_\nu(k_n r_j) e^{\mathrm{i}\nu\theta_j}, \tag{3}$$

with discrete coefficients $D_{n\nu}^j$. In these expansions, I_ν and K_ν denote the modified Bessel functions of the first and second kind, respectively, both of order ν.

2.2 Diffraction transfer operators

In what follows, we make extensive use of diffraction transfer operators, sometimes referred to as T-matrices. In general, it is possible to relate the total incident and scattered partial waves for any structure through the diffraction characteristics of that body in isolation. There exist diffraction transfer operators B^l that relate the coefficients of the incident and scattered partial waves, such that

$$A_{m\mu}^l = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} B_{mn\mu\nu}^l D_{n\nu}^l, \tag{4}$$

where A^l are the amplitudes of the scattered modes due to the incident modes of amplitude D^l. The idea of the diffraction transfer operator is not restricted to a single structure. We can thus associate such an operator with a module.

3 Scattering by a module

We first present an efficient method to calculate the diffraction transfer operator for a module of bodies, which we will group in a periodic line array in the next section.

The scattering properties of a finite number of bodies can be calculated in many different ways, for example by using the finite element method, which involves discretising all body surfaces, or, more efficiently, using an interaction theory (Kagemoto & Yue, 1986; Peter & Meylan, 2004). For our purposes, the scattered wavefield needs to be represented in terms of eigenfunction expansions (2), (3) in order to allow the scattering properties of a module to be described by a single diffraction transfer operator. We briefly summarise how this can be achieved using the interaction theory. Note that the idea of using the interaction theory to group several bodies into modules has been successfully applied previously in the context of interactions of finitely many bodies (Kashiwagi, 2000; Chakrabarti, 2000).

A system of equations for the unknown coefficients (in the expansion (2)) of the scattered wavefields of all N bodies of the module can be developed based on transforming the scattered potential of Δ_j into an incident potential upon

Δ_l $(j \neq l)$. Doing this for all bodies simultaneously, and relating the incident and scattered potential for each body, a system of equations for the unknown coefficients follows.

It turns out that the coefficients of the scattered wavefield of each body Δ_l in the expansion (2) satisfy

$$A^l_{m\mu} = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} B^l_{mn\mu\nu} \Big[\hat{D}^l_{n\nu} \tag{5}$$

$$+ \sum_{\substack{j=1 \\ j\neq l}}^{N} \sum_{\tau=-\infty}^{\infty} A^j_{n\tau} (-1)^\nu K_{\tau-\nu}(k_n R_{jl}) \mathrm{e}^{\mathrm{i}(\tau-\nu)\vartheta_{jl}} \Big],$$

$m \in \mathbb{N}$, $\mu \in \mathbb{Z}$, $l = 1, \ldots, N$, where (R_{jl}, ϑ_{jl}) are the coordinates of the mean-centre position of the lth body in terms of the coordinate system of the jth body and $\hat{D}^l_{n\nu}$ are the coefficients of the incident wave in the expansion (3) centred at the lth body. Solving the system of equations (5) for all possible cylindrical incident waves of unit amplitude (i.e. for $D_{n\nu} = 1$ for one (n, ν) at a time and zero for the others) and adding up the scattered waves with respect to the origin yields the diffraction transfer operator M of the module made up of the bodies Δ_j, $j = 1, \ldots, N$.

4 Scattering by a periodic array

In the same way as in the previous section, we can use the interaction theory to derive a system of equations for the periodic line array made up of identical modules with diffraction transfer operator M, where the module mean-centre positions are located at $(jR, 0)$, $j \in \mathbb{Z}$ (see Peter *et al.*, 2006, for details).

Owing to the periodicity of the geometry and of the incident wave, the coefficients $A^l_{m\mu}$ can be written as $A^l_{m\mu} = P_l A^0_{m\mu} = P_l A_{m\mu}$, say, where $P_l = \mathrm{e}^{\mathrm{i}lRk\cos\chi}$. The same can be done for the coefficients of the incident ambient wave, i.e. $\tilde{D}^l_{n\nu} = P_l \tilde{D}_{n\nu}$, where $\tilde{D}_{n\nu}$ are the coefficients of ϕ^{In} in the expansion (3). Noting that $P_l^{-1} = P_{-l}$ and $P_j P_l = P_{j+l}$, (5) simplifies to

$$A_{m\mu} = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} M_{mn\mu\nu}$$

$$\times \Big[\tilde{D}_{n\nu} + (-1)^\nu \sum_{\tau=-\infty}^{\infty} A_{n\tau}\, \sigma^n_{\tau-\nu} \Big], \tag{6}$$

where the constants σ^n_ν are defined as $\sigma^n_\nu = \sum_{j=1}^{\infty} (P_{-j} + (-1)^\nu P_j) K_\nu(k_n jR)$. These can be evaluated separately since they do not contain any unknowns. The efficient computation of the constants σ^0_ν is not trivial but appropriate methods are outlined in Peter *et al.* (2006) based on results of Linton (1998).

4.1 The far field

In this section, we summarise how the far field can be calculated, which describes the scattering far away from the array. First, we define the scattering angles, which give the directions of propagation of plane scattered waves far away from the array. Letting $p = 2\pi/R$, the scattering angles χ_m are

$$\chi_m = \arccos(\psi_m/k), \text{ where } \psi_m = k\cos\chi + mp,$$

and we write ψ for ψ_0. Also note that $\chi_0 = \chi$ by definition. If $|\psi_m| < k$, we say that $m \in \mathcal{M}$ and then $0 < \chi_m < \pi$.

It turns out that, as $y \to \pm\infty$, the far field consists of a set of plane waves propagating in the directions $\theta = \pm\chi_m$:

$$\phi \sim \phi^{\mathrm{In}} + f_0(z) \sum_{m\in\mathcal{M}} A^\pm_m \mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\chi_m)}, \tag{7}$$

where

$$A^\pm_m = \frac{\pi \mathrm{i}}{kR} \frac{1}{\sin\chi_m} \sum_{\mu=-\infty}^{\infty} A_{0\mu}\, \mathrm{e}^{\pm\mathrm{i}\mu\chi_m}. \tag{8}$$

It is implicit in the above that $\sin\chi_m \neq 0$ for all m and we do not consider the resonant case of equality here.

4.2 Reflection and transmission matrices

For given k, R and χ, the far-field scattering characteristics of a line array L^j are completely described by the reflection and transmission matrices $\mathfrak{r}^j, \mathfrak{t}^j \in \mathbb{C}^{\#\mathcal{M}\times\#\mathcal{M}}$, in which the coefficients A^-_m and $\delta_{m0} + A^+_0$, respectively, are saved, calculated for each incident angle χ_m.

It is useful to know how the reflection and transmission matrices change if the array undergoes a translation such that the mean-centre position of the zeroth body (originally located at $(0, 0)$) is shifted to lie at some new position (x, y). Writing $P = \lceil \exp(\mathrm{i}kx\cos\chi_m) \rfloor$ and $Q = \lceil \exp(\mathrm{i}ky\sin\chi_m) \rfloor$, where $\lceil a_m \rfloor$ is a diagonal matrix with diagonal elements a_m, the reflection and transmission matrices of the translated array are given by $P^{-1}Q\mathfrak{r}^jQP$ and $P^{-1}Q^{-1}\mathfrak{t}^jQP$, resp. As expected, a shift in the x-direction by a multiple of the array spacing R leaves the matrices unchanged.

5 Scattering by multiple arrays

It was found in Peter *et al.* (2006) that the far-field approximation is very good even near the line array and numerical experiments confirm this observation. Based on ideas of McPhedran *et al.* (1999) for electromagnetic scattering by cylinders, we present an explicit iterative method to stack up many periodic line arrays.

Once k, R and χ are fixed, a plane wave of incident angle χ_l for a $l \in \mathcal{M}$ will results in transmitted and reflected waves travelling in the directions $\pm\chi_m$, $m \in \mathcal{M}$. Thus, for a stack of arrays of the same horizontal spacing R, only waves in the directions $\pm\chi_m$, $m \in \mathcal{M}$, need to be considered.

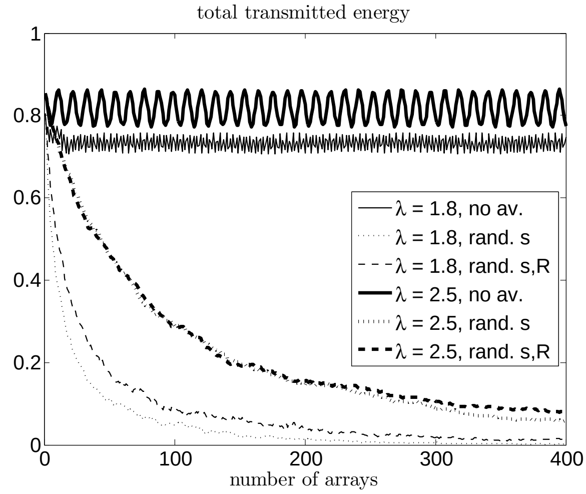
For given k, R and χ, the scattering characteristics of a line array L^j are completely described by the matrices $\mathfrak{r}^j, \mathfrak{t}^j \in \mathbb{C}^{\#\mathcal{M}\times\#\mathcal{M}}$. Assuming that the reflection and transmission matrices for a stack of $n-1$ modules is already known and given by $\mathfrak{R}_{n-1}$ and $\mathfrak{T}_{n-1}$, the nth array can be added on as follows: Let $s_n > 0$ be the (vertical) spacing between the stack and the array to be added on. With the diagonal matrix $Q_n = \lceil \exp(\mathrm{i}ks_n \sin\chi_m)\rfloor$ (cf. §4.2), the total reflection and transmission matrices of the stack composed of n arrays is given by

$$\begin{aligned} \mathfrak{R}_n &= \mathfrak{r}_n + \mathfrak{t}_n Q_n \mathfrak{R}_{n-1} Q_n (\mathrm{I} - \mathfrak{r}_n Q_n \mathfrak{R}_{n-1} Q_n)^{-1} \mathfrak{t}_n, \\ \mathfrak{T}_n &= Q_n^{-1} \mathfrak{T}_{n-1} Q_n (\mathrm{I} - \mathfrak{r}_n Q_n \mathfrak{R}_{n-1} Q_n)^{-1} \mathfrak{t}_n. \end{aligned} \tag{9}$$

6 Simulation results

We present a typical set of results, which can be extracted from the presented theory. We consider square ice floes having side length 2 and stiffness $\beta = 0.02$, mass $\gamma = 0.02$ (in dimensionless parameters of Meylan, 2002) and Poisson's ratio $\nu = 1/3$. We take $d = 1/2$ and $\chi = \pi/3$ and consider two incident wavelengths: $\lambda = 1.8$ and $\lambda = 2.5$.

In the figure below, the total transmitted energy is plotted versus the number of arrays in the stack for arrays with spacing $R = 4$ and $s_n = 4$. Furthermore, results are presented for an average of 200 simulations, where s_n randomly changes about mean 4 with standard deviation 1/3 and where R and s_n randomly change about mean 4. It can be seen that some randomisation is necessary in order to get rid of certain resonances introduced by the periodicity. The curves show a clear exponential type attenuation of wave energy, exactly as measured, and we believe that we can extract the relevant geophysical parameters straightforwardly using our solution method.



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