Resonances and the Approximation of Wave Forcing for Elastic Floating Bodies

Michael H. Meylan[1] **Marko Tomic**[2]

[1]**Department of Mathematics, University of Auckland, New Zealand**

[2]**Faculty of Mech. Eng. and Naval Architecture, University of Zagreb, Croatia**

e-mail addresses: meylan@math.auckland.ac.nz, marko.tomic@fsb.hr

1 Introduction

The central problem in *hydroelasticity* is to calculating the response of an elastic body subject to wave forcing. The majority of the research in hydroelasticity has been aimed at predicting the response in the frequency domain, and details of the various methods developed can be found in the review articles (Squire *et al.*, 1995; Squire, 2007), which are mainly focused on ice, and (Kashiwagi, 2000; Watanabe *et al.*, 2004), which are focused primarily on *VLFS*.

We present here a method to determine approximately the positions of the *resonances* or *scattering frequencies* for a floating elastic body in a fluid with a free surface. Resonances are rather mysterious objects which are far from fully understood. The simplest way to think of them is as representing the new positions of the eigenvalues of the free vibration of the elastic body in the presence of the fluid. Associated with these points are functions which represent the modes of vibration of the body. However, they are not eigenvalues of any operator and the associated functions are not eigenvectors. The expansion of the solution in terms of the resonances will generally be only an approximation. One exception to this is the problem of an elastic plate on shallow water (Meylan, 2002), where the resonances do give a complete solution. Resonances have been investigated in the water-wave context by Hazard & Lenoir (1993, 2002) and for the case of a plate by Hazard & Loret (2007); Peter & Meylan (2008). They have also been found for non-elastic bodies, and appear in Evans & Porter (1997); McIver (2005); Meylan & Eatock Taylor (2009).

In this present paper we begin with the general equation in the frequency domain, written in terms of the generalized modes. This equation is closely related to the standard method to solve for a rigid body in terms of the six rigid-body motions. The resonances arise when we extend the definition of the added mass and damping analytically for complex frequencies. We present here a method to approximate the positions of the resonances, using only values for the added mass and damping calculated on the real axis. This means that the method could be used in commercial code without requiring modification to work for complex frequencies. We also show how knowledge of the resonances allows us to approximate the response for real frequencies.

We apply the theory to the problem of a two-dimensional elastic plate floating on the water surface and present some numerical results. We show that we get good results using this theory. We also show that the method becomes less valid as the relative difference between the plate and fluid densities becomes greater.

2 General Equations for an Elastic Body

We do not derive here the general equations for the solution in the frequency domain for a floating elastic body, but rather assume them. The equations take the form

$$\left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{C} - \omega^2\mathbf{A}(\omega) + \mathrm{i}\omega\mathbf{B}(\omega)\right)\xi = \mathbf{f}, \tag{1}$$

where ω is the radian frequency, $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the hydrostatic restoring matrix, $\mathbf{A}$ is the added mass matrix, $\mathbf{B}$ is the damping matrix, ξ is the vector of generalized modes and f is the forcing due to the incident wave. To simplify this equation we introduce the notation $\alpha = \omega^2$ and also define

$$\mathbf{\Phi}(\alpha) = -\omega^2\mathbf{A}(\omega) + \mathrm{i}\omega\mathbf{B}(\omega). \tag{2}$$

3 Associate Eigenvalue Type Equations

Associated with equation (1) are certain equations which define either eigenvalues or resonances. Some of these are well-known, but we repeat the definition here for the purpose of illustrating the important difference between wet modes and resonances. The first equation we begin with is given by

$$(\mathbf{K} - \alpha\mathbf{M})\,\xi = 0, \tag{3}$$

which gives the free modes of vibration for the elastic body. The second equation is

$$(\mathbf{K} - \alpha\mathbf{M} + \mathbf{C})\,\xi = 0, \tag{4}$$

which gives the wet modes. The third equation is the defining equation for a resonance,

$$(\mathbf{K} - \alpha\mathbf{M} + \mathbf{C} + \mathbf{\Phi}(\alpha))\,\xi = 0. \tag{5}$$

Equation (5) has no solutions for real α. However we may extend the definition of $\mathbf{\Phi}$ by analytic continuation to complex α. This analytic extension is generally nothing more than computing $\mathbf{\Phi}$ for complex α, using the same formula as for real α.

4 Approximate Solution for Resonances

Most software which calculates the added mass and damping only works for real α, and we develop here a theory to find complex resonances using only the solution for real α. The approximation theory is also useful for calculating the exact position of the resonances. The approximation is based on a Taylor series expansion of the matrix $\mathbf{\Phi}$. This matrix is analytic, which means that the derivative in the real direction is the same as the derivative in any non-real direction, and hence we can estimate the complex values from knowledge of the real derivative. We write $\alpha = \alpha_0 + \delta$ and write equation (5) as

$$(\mathbf{K} - (\alpha_0 + \delta)\mathbf{M} + \mathbf{C} + \mathbf{\Phi}(\alpha_0) + \delta\mathbf{\Phi}')\,\xi = 0, \tag{6}$$

where $\mathbf{\Phi}'$ is the derivative, which can be calculated relatively easily numerically. This is then a standard matrix equation in δ which can be solved. The solution for small δ can be expected to give the approximate solution for the resonance. Note that this approximation is only valid close to α_0, and also that we have no way of determining the region in which this approximation is valid.

We denote the eigenvector associated with the smallest value of δ by $\mathbf{u_n}$ and the associated resonance value by $\mu_n = \alpha_0 + \delta$. Note that the equation satisfied by $\mathbf{u_n}$ is

$$(\mathbf{K} - \alpha_0\mathbf{M} + \mathbf{C} + \mathbf{\Phi}(\alpha_0))\,\mathbf{u_n} = (\delta\mathbf{M} - \delta\mathbf{\Phi}')\,\mathbf{u_n}. \tag{7}$$

5 Approximation of the Response

We can estimate the solution of equation (1) using the approximate resonances. The critical formula is that the solution can be written as

$$\xi(\alpha) = \sum_n \mathbf{u_n} \frac{\mathbf{u_n}.\mathbf{f}}{(\alpha - \mu_n)\mathbf{u_n}.\,(\mathbf{M} - \mathbf{\Phi}')\,\mathbf{u_n}}. \tag{8}$$

6 Floating Elastic Plate

The two-dimensional floating elastic plate of finite length is the simplest and best-studied problem in hydroelasticity. Solutions in the frequency domain were first presented by Meylan & Squire (1994); Newman (1994). It is the most logical place to start investigating the present theory. We assume that the plate occupies the region $(-L, L)$ and that the water is of constant depth h. The equations for a floating elastic plate are

$$\Delta\phi = 0,\ -h < z < 0, \tag{9a}$$

$$\partial_z\phi = 0,\ z = -h, \tag{9b}$$

$$\omega^2\phi = \partial_z\phi,\ x \notin (-L, L),\quad z = 0, \tag{9c}$$

$$\mathrm{i}\omega\sum_{n=0}^{\infty}\xi_n w_n = \partial_z\phi,\ x \in (-L, L),\ z = 0, \tag{9d}$$

$$\sum_{n=0}^{\infty}\xi_n\left(1 + \beta\lambda_n^4\right)w_n - \omega^2\gamma\sum_{n=0}^{\infty}\xi_n w_n \tag{9e}$$

$$= -\mathrm{i}\omega\phi,\ x \in (-L, L),\ z = 0, \tag{9f}$$

where w_n are the free modes of vibration of an elastic beam. In this case the stiffness matrix is given by

$$\mathbf{K} = \lceil\beta\lambda_n^4\rfloor, \tag{10}$$

where $\lceil \ldots \rfloor$ denotes a diagonal matrix. The mass matrix is given by

$$\mathbf{M} = \gamma\mathbf{I}, \tag{11}$$

where $\mathbf{I}$ is the identity matrix. The hydrostatic restoring matrix is given by

$$\mathbf{C} = \mathbf{I}. \tag{12}$$

We do not discuss the calculation of $\mathbf{\Phi}$ here.

7 Results

We present results for the case of a floating elastic plate, with $L = 5$ and $h = 2$. We consider four values for β and γ, $\beta = \gamma = 1$, $\beta = \gamma = 0.5$, $\beta = \gamma = 0.2$, and $\beta = \gamma = 0.1$. Figure 1 shows the reflection coefficient and the resonances for these four cases, with $\times$ representing the approximate position and $\circ$ representing the exact position of the resonances, calculated by a method which we do not discuss here. We can see that the approximate solution for the resonance is close to exact resonance, except for the lowest resonances. It is interesting to note that the resonances are associated with a peak in the energy and also with a zero in the reflection coefficient. No explanation for the zero in reflection is known to the authors. Also note there is not necessarily exact matching between the zero in reflection and resonance. There is also a further resonances which occurs for negative real part of α, which cannot be found accurately by the approximate method and this is not shown. The resonances have a correspondence with the free vibrational modes and the response is dominated by, but does not consist solely of, this mode. The properties of symmetry and anti-symmetry of the modes are preserved by the resonances.

Figure 2 shows the potential energy in the plate averaged over one period for the exact (solid line) and approximate solution (dashed line) using equation (6). The energy is given by

$$\sum \lambda_n^4 |\xi_n|^2. \tag{13}$$

Note that there is no bending energy in the two lowest modes (which are the rigid body modes) and for this reason any errors in estimating these modes are not apparent in this figure. It is clear from this figure that the higher that β and γ are the more accurate is the approximate solution, and also that the approximate solution is more accurate for higher frequencies.

8 Summary

We have presented a method to calculate the resonances for a floating body using the solution for real frequencies. This method is applicable to any floating body, however it seems likely that it will be most effective for elastic bodies. We have presented a numerical investigation of this method for the case of a two-dimensional floating plate of negligible submergence (the simplest hydroelastic problem). The results for this case are very encouraging and we believe that the theory could easily be extended to other elastic structures, especially to the hydroelastic response of very large container ships which was the motivation for this research.

References

Evans, D. & Porter, R. 1997 Near-trapping of waves by circular arrays of vertical cylinders. *Appl. Ocean Res.* **19** (2), 83–99.

Hazard, C. & Lenoir, M. 1993 Determination of scattering frequencies for an elastic floating body. *SIAM J. Math. Anal.* **24** (4), 1458–1514.

Hazard, C. & Lenoir, M. 2002 Surface water waves. In *Scattering* (ed. eds. R. Pike & P. Sabatier), pp. 618–636. Academic.

Hazard, C. & Loret, F. 2007 Generalized eigenfunction expansions for scattering problems with an application to water waves. *Proc. Roy. Soc. Edinburgh* **137A**, 995–1035.

Kashiwagi, M. 2000 Research on hydroelastic response of VLFS: Recent progress and future work. *Int. J. Offshore Polar Engineering* **10** (2), 81–90.

McIver, P. 2005 Complex resonances in the water-wave problem for a floating structure. *J. Fluid Mech.* **536**, 423–3257.

Meylan, M. H. 2002 Spectral solution of time dependent shallow water hydroelasticity. *J. Fluid Mech.* **454**, 387 – 402.

Meylan, M. H. & Eatock Taylor, R. 2009 Time-dependent water-wave scattering by arrays of cylinders and the approximation of near trapping. *J. Fluid Mech.* In press.

Meylan, M. H. & Squire, V. A. 1994 The response of ice floes to ocean waves. *J. of Geophysical Research* **99** (C1), 891–900.

Newman, J. N. 1994 Wave effects on deformable bodies. *Appl. Ocean Res.* **16**, 45 – 101.

Peter, M. A. & Meylan, M. H. 2008 Time-dependent interaction of water waves and a vertical elastic plate. In *Proc. 23rd Int. Workshop on Water Waves and Floating Bodies, Jeju, Korea* (ed. H. S. Choi & Y. Kim), pp. 144–147.

Squire, V. A. 2007 Of ocean waves and sea-ice revisited. *Cold Regions Science and Technology* **49** (2), 110–133.

Squire, V. A., Dugan, J. P., Wadhams, P., Rottier, P. J. & Liu, A. J. 1995 Of ocean waves and sea ice. *Annu. Rev. Fluid Mech.* **27**, 115–168.

Watanabe, E., Utsunomiya, T. & Wang, C. M. 2004 Hydroelastic analysis of pontoon-type VLFS: a literature survey. *Eng. Struct.* **26** (2), 245–256.

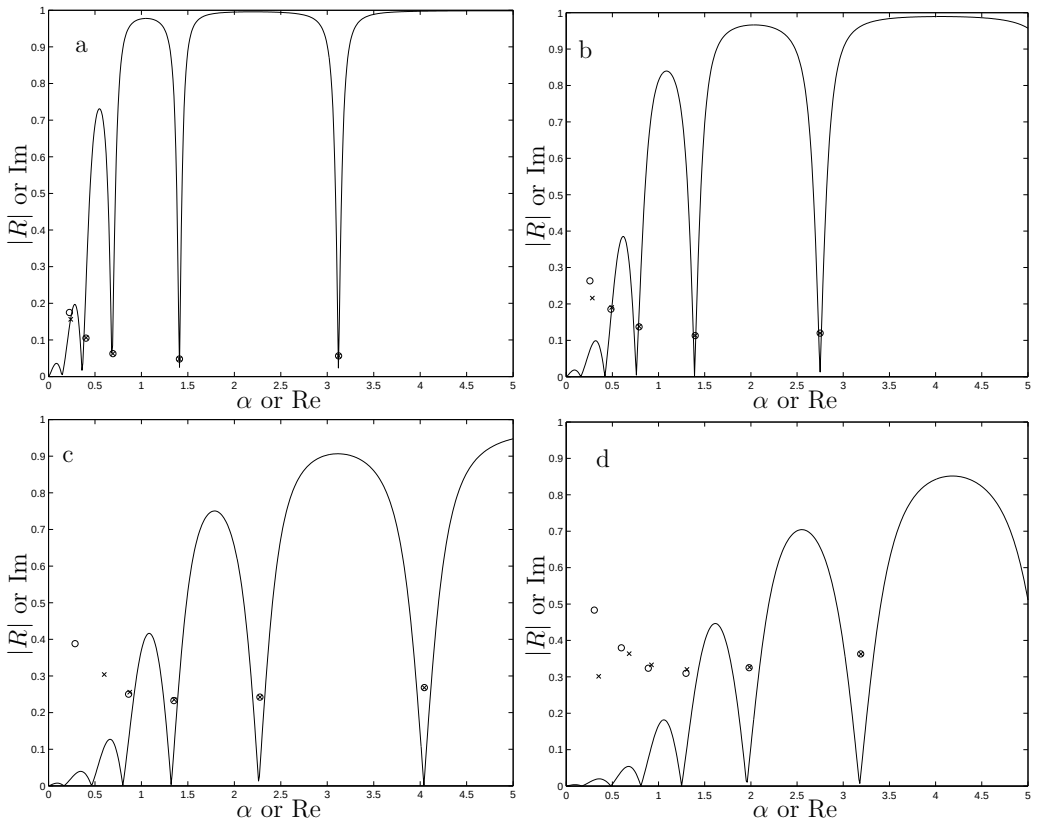

Figure 1: $|R|$ versus α and position of the resonances ($\times$ approximate and $\circ$ exact) in the complex plane. $L = 5$, $h = 2$, $\beta = \gamma = 1$ (a), $\beta = \gamma = 0.5$ (b), $\beta = \gamma = 0.2$ (c), and $\beta = \gamma = 0.1$ (d).

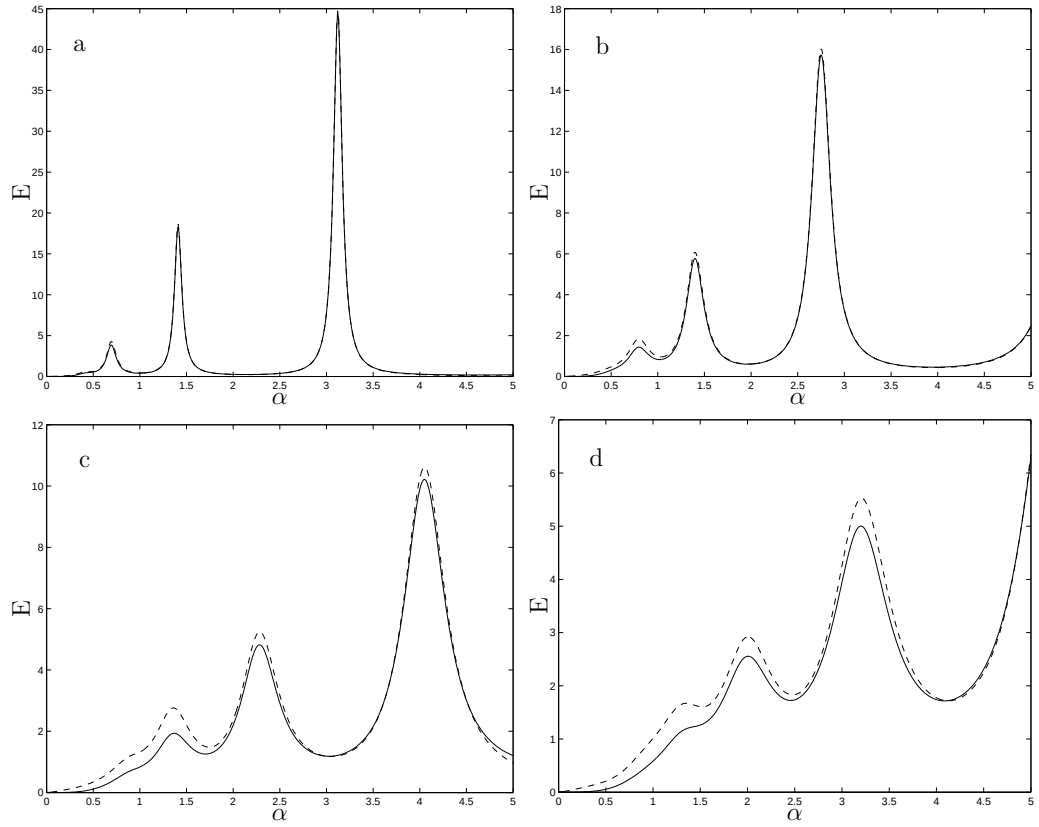

Figure 2: Bending energy as a function of α for the exact solution (solid line) and the approximate solution using equation (6). $L = 5$, $h = 1$, $\beta = \gamma = 1$ (a), $\beta = \gamma = 0.5$ (b), $\beta = \gamma = 0.2$ (c), and $\beta = \gamma = 0.1$ (d).