‘High spots’ of the free surface for the fundamental sloshing mode

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The present note deals with the sloshing problem, two- and three-dimensional versions of which describe free oscillations of water in an infinitely long canal of uniform cross-section and in a container, respectively. The case of infinitely deep water is treated in both dimensions as well. Our aim is to study the location of ‘high spots’ on the free surface for the fundamental mode in various water domains. We recall that the free-surface elevation is proportional to the trace of mode’s velocity potential taken on the undisturbed level, and so at every moment the location of high spots is determined by the projections on this level of trace’s maxima (defined up to a constant non-zero factor). Moreover, both maxima and minima of the trace must be considered when a time-harmonic factor ($\cos \omega t$ or $\sin \omega t$, where ω is the radian frequency of water oscillations) is removed. Indeed, they give the high spots during the complementary time intervals when this factor is taken into account.

The sloshing problem was the subject of a great number of studies (see [5] for a review). Among recent works we mention the papers [7] and [8], where the question of simplicity of sloshing frequencies was investigated along with some other properties of these frequencies and the corresponding eigenfunctions which are either velocity potentials or stream functions describing free oscillation modes. The present considerations essentially rely upon some results obtained in [7] and [8].

1 Two-dimensional problem for various water domains

Let an inviscid, incompressible, heavy fluid (water) occupy a canal; its cross-section W is assumed to be a two-dimensional, bounded, simply connected domain without cusps on the piecewise smooth boundary ∂W, which consists of the cross-sections of the free surface and of the bottom denoted by F and $B = \partial W \setminus \bar{F}$, respectively. The former has a finite width and is represented by an interval of the x-axis in appropriate Cartesian coordinates (x, y) with the y-axis directed upwards; the latter is the union of open arcs that lie in the half-plane $y < 0$ and are complemented by the corner points (if there are any) connecting these arcs. The surface tension is neglected and the water motion is supposed to be two-dimensional, irrotational, and of small-amplitude. These assumptions lead to the following boundary value problem for the velocity potential $u(x, y)$ with a time-harmonic factor removed:

$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \quad u_y = \nu u \quad \text{on } F, \quad \partial u/\partial n = 0 \quad \text{on } B, \quad \int_F u(x, 0)\, \mathrm{d}x = 0. \tag{1}$$

The last condition is imposed to exclude the zero eigenvalue; the spectral parameter ν is equal to ω^2/g, where g is the acceleration due to gravity. Generally speaking, the boundary ∂W has corner points, and so the condition $\int_W |\nabla u|^2\, \mathrm{d}x\mathrm{d}y < \infty$ must be added to relations (1) to avoid strong singularities near these points.

The formulated problem has a discrete spectrum; that is, there exists a sequence of eigenvalues $0 < \nu_1 < \nu_2 \leq \nu_3 \leq \ldots \leq \nu_n \leq \ldots$, each having a finite multiplicity equal to the number of repetitions. Moreover, $\nu_n \to \infty$ as $n \to \infty$, and the fundamental eigenvalue ν_1 is simple ($\nu_1 < \nu_2$). The latter fact is proved in [7] along with the assertion that the eigenfunction u_1 has only one nodal line connecting $\bar{F}$ and $\bar{B}$. The proofs are based on a non-local variational principle for another equivalent statement of the problem, in which the following conditions

$$v_{xx} + v_{yy} = 0 \quad \text{in } W, \quad -v_{xx} = \nu v_y \quad \text{on } F, \quad v = 0 \quad \text{on } B \tag{2}$$

are used instead of (1). Here v is the conjugate to u harmonic function (stream function). It is possible to choose v_1 so that it is positive on $W \cup F$ and this fact plays an essential role in proving the following assertion that describes the location of high spots for the fundamental sloshing mode.

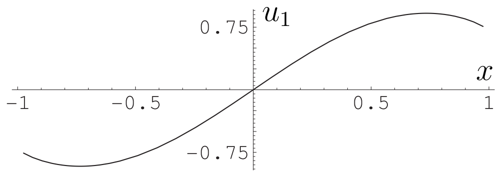

Figure 1: The trace $u_1(x, 0)$ for the ice-fishing problem with $b = 0$; $x \in [-0.99, 0.99]$.

PROPOSITION 1.1. *Let $\bar{B}$ be the graph of a function given on $\bar{F}$. Then the traces of u_1 on $\bar{F}$ and $\bar{B}$ are monotonic functions (say, both increasing) of the corresponding arguments which are x on $\bar{F}$ and the arc length s measured form the left end-point on $\bar{B}$. Moreover, $\max_{(x,y)\in\bar{W}} u_1(x, y)$ and $\min_{(x,y)\in\bar{W}} u_1(x, y)$ are attained at the end-points of F.*

COROLLARY 1.2. *The trace of u_1 is monotonically increasing on F if and only if the trace of v_1 is a concave function on F attaining its maximum at the end-point of the nodal line of u_1.*

By the definition of high spots, each end-point of F gives the high spot of the free surface for the fundamental mode during the time intervals defined by a time-harmonic factor. The fact that high spots are the end-points of F resembles the so-called 'hot spots' conjecture which was formulated by Rauch in 1974 (see, e.g., [1]). The conjecture says that any eigenfunction corresponding to the smallest non-zero eigenvalue of the Neumann Laplacian in a domain $D \subset \mathbb{R}^d$ attains its maximum and minimum values on ∂D. During the past decade, the hot spots conjecture has been intensively studied (see [3] for a survey). It was proved for some classes of two- and three-dimensional domains, but is still an open question for an arbitrary convex two-dimensional domain. On the other hand, there exists a multiply connected domain that serves as a counterexample to the conjecture (see, e.g., [4]). We show (see Proposition 2.1 below) that the hot spots conjecture is equivalent to the question about high spots for the 'glass' sloshing problem.

Let us turn to the so-called ice-fishing problem (also referred to as the infinite-dock problem). For infinitely deep water when the cross-section of the water domain is $W = \mathbb{R}^2_- = \{x \in \mathbb{R},\ y < 0\}$, we consider $F = \{b < |x| < b + 1, y = 0\}$, where b is a non-negative parameter. For $b = 0$ the problem coincides with that for $F = \{|x| < 1, y = 0\}$. The fact that the total length of the free surface is normalised to be equal to two is not a restriction in view of domain similarity. Again the condition $\int_{\mathbb{R}^2_-} |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y < \infty$ must be added to relations (1).

PROPOSITION 1.3. *Let b either vanish or be sufficiently large. Then the points, where the fundamental eigenfunction of the ice-fishing problem attains its maximum and minimum values, are inner points of F symmetric with respect to zero.*

The case $b = 0$ is illustrated in Figure 1 and the proof for this case is based on the asymptotic formula describing the behaviour of u_1 in a neighbourhood of the dock tip $(1, 0)$ (see [8], formula 2.1). For sufficiently large b, the proof follows from the asymptotic formula valid for $u_1(x, 0)$ as $b \to \infty$ (see [8], Theorem 3.1).

Comparing the results of Propositions 1.1 and 1.3, we are in a position to formulate the following

CONJECTURE 1.4. Let W be a bounded domain with smooth B such that at least one angle between B and F is greater than $\pi/2$. Then the fundamental eigenfunction u_1 attains at least one of its extremum values at an inner point of F.

2 Proof of Proposition 1.1 (an outline)

Our proof of Proposition 1.1 is based on two assertions that involve another mixed Steklov problem that distinguishes from the sloshing problem only by the boundary condition on B, namely:

$$w_{xx} + w_{yy} = 0 \ \text{ in } W, \quad w_y = \lambda w \ \text{ on } F, \quad w = 0 \ \text{ on } B. \tag{3}$$

This problem also has discrete spectrum $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$, and the eigenvalue λ_1 is simple. The corresponding eigenfunctions w_n, $n = 1, 2, \ldots$, belong to the subspace $H^1_B(W)$ of the Sobolev space $H^1(W)$ ($H^1_B(W)$ consists of functions in $H^1(W)$ that vanish on B); the traces $w_n(x,0)$ form an orthogonal basis in $L^2(F)$. Arguing in the same way as in the proof of Theorem 3.8, [2], one arrives at the following domain monotonicity property similar to that holding for the fundamental eigenvalue of the Dirichlet Laplacian.

LEMMA 2.1. *Let R be a simply connected subdomain of W such that $R \neq W$ and $F \cap \partial R$ is an open non-empty set. Then $\lambda_1^R > \lambda_1^W$, where the superscript indicates the domain in which problem* (3) *is considered.*

Combining this lemma and the well-known monotonicity property valid for the fundamental eigenvalue of the sloshing problem (see, e.g., [9]), one obtains the first assertion of

PROPOSITION 2.2. *Let W be confined to the semi-strip $\{|x| < a, y < 0\}$. Then $\nu_1 \leq \lambda_1$ and the equality holds only for the infinitely deep rectangular domain $\{|x| < a, -\infty < y < 0\}$.*

Note that for the infinitely deep rectangular domain we have:

$$\nu_1 = \lambda_1 = \frac{\pi}{2a}, \quad \text{whereas } u_1(x,y) = \sin\frac{\pi x}{2a}\exp\frac{\pi y}{2a} \text{ and } w_1(x,y) = \cos\frac{\pi x}{2a}\exp\frac{\pi y}{2a},$$

which proves the second assertion.

Proof of Proposition 1.1. For the sake of brevity we omit the subscript 1 at u and v which are the fundamental eigenfunctions of problems (1) and (2), respectively. Since v is positive on $W \cup F$ and B is the graph of a function given on F, we have that $v_y \geq 0$ on B. In view of the second boundary condition in problem (1), this implies $u_x = \frac{\partial u}{\partial t} t_x \geq 0$ on B, where $\frac{\partial}{\partial t}$ denotes the tangential derivative and t_x is the projection of the unit tangent on the x-axis (the direction of tangent coincide with the increasing of the arc length s on B). Note that t_x is non-negative because B is the graph, and so the same is true for $\frac{\partial u}{\partial t}$, which proves the assertion that the trace of u increases on B.

In order to show that $u_x \geq 0$ on F, which proves the first assertion, we assume the contrary, i.e., that u_x changes sign at some inner point of F. Then there exists an open nonempty subset of W such that $u_x < 0$ on it; let R be a connected component of this subset. Note that $u_x = 0$ on $\partial R \cap B$ because $u_x \geq 0$ on B and $u_x < 0$ on R. It follows that u_x satisfies the boundary value problem (3) on the domain R. Moreover, $\lambda_1^R = \nu_1^W$ in the second condition (3) for u_x, which one obtains differentiating the second condition (1) with respect to x. By Lemma 2.1 we get $\lambda_1^W < \lambda_1^R = \nu_1^W$ which contradicts Proposition 2.2 since W is not the infinitely deep rectangular domain. Hence u_x does not change sign on F.

3 Three-dimensional problem for various water domains

We consider two types of water domains. A domain of the first type is

$$W = \{x = (x_1, x_2) \in D,\ y \in (-d, 0)\},$$

where D is a bounded two-dimensional domain and $d \in (0, \infty]$; i.e., the depth of container is constant and its walls are vertical. In this case $F = D \times \{y = 0\}$, $B = \partial W \setminus \bar{F}$, and the sloshing problem is referred to as the 'glass' problem. The domain of the second type is

$$W = \mathbb{R}^3_- = \{x = (x_1, x_2) \in \mathbb{R}^2,\ y \in (-\infty, 0)\}$$

with $F = \{r < 1,\ y = 0\}$, $r^2 = x_1^2 + x_2^2$, and $B = \partial\mathbb{R}^3_- \setminus \bar{F}$. This problem is called the ice-fishing problem (the fact that hole's radius is equal to one is not a restriction in view of domain similarity). The Laplace equation in W takes the form $\nabla_x^2 u + u_{yy} = 0$, where $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2)$. The last three conditions (1) must hold for both problems as well as the condition $\int_W |\nabla u|^2 \,\mathrm{d}x\mathrm{d}y < \infty$.

Separating the y-variable in the glass problem as follows: $u(x,y) = \phi(x)\cosh k(y+d)$ for $d < \infty$ and $\phi(x)\,\mathrm{e}^{ky}$ when $d = \infty$, one obtains that ϕ is an eigenfunction of the problem:

$$\nabla_x^2\phi + k^2\phi = 0 \ \text{ in } D, \quad \partial\phi/\partial n_x = 0 \ \text{ on } \partial D, \ \int_D \phi \,\mathrm{d}x = 0,$$

where $\nu = k \tanh kd$ for $d < \infty$ and $\nu = k$ when $d = \infty$. Hence we arrive at

PROPOSITION 2.1. *A point $(x, 0) \in \partial F$ is a high spot for the glass problem if and only if $x \in \partial D$ is a hot spot for the Neumann Laplacian in D.*

Let us turn to the ice-fishing problem investigated in [6]. In particular, it was proved that the eigenfunctions $\psi_1(r, y)\frac{x_i}{r}$, $i = 1, 2$, correspond to the fundamental eigenvalue ν_1. Here

$$\psi_1(r, y) = \nu_1 \int_0^1 \psi_1(s, 0)\, s\, \mathrm{d}s \int_0^\infty J_1(kr)\, J_1(ks)\, \mathrm{e}^{ky}\, \mathrm{d}k,$$

and the trace of ψ_1 on F is the fundamental eigenfunction of

$$\psi(r, 0) = \nu \int_0^1 \psi(s, 0)\, s\, I(r, s)\, \mathrm{d}s, \quad \text{where } I(r, s) = \int_0^\infty J_1(kr)\, J_1(ks)\, \mathrm{d}k, \;\; r \in (0, 1),$$

and J_1 is the Bessel function. The fundamental eigenvalue of the positive kernel $I(r, s)$ is also equal to ν_1. Choosing $\psi_1(r, 0)$ to be positive for $r \in (0, 1)$ (it is clear that $\psi_1(0, 0) = 0$ because $J_1(0) = 0$) and such that $\psi_1(1, 0) = 1$, we have that the asymptotic formula (see Proposition 4, [6]):

$$\psi_1(r, 0) = 1 + \frac{\nu_1}{\pi}(r - 1)\, \log|r - 1| + O\left(|r - 1|\right)$$

is valid as $r \to 1$. This formula can be differentiated, thus giving that $\frac{\partial \psi_1}{\partial r}(r, 0) \to -\infty$ as $r \to 1$. Now taking into account the formula for eigenfunctions of the ice-fishing problem, we arrive at

PROPOSITION 2.2. *Both fundamental eigenfunctions of the ice-fishing problem for a single circular hole attain their maximum and minimum values at inner points of F.*

The simplest way to generalise Proposition 1.1 to the three-dimensional case is to consider a domain W with rotational symmetry (a 'wine glass'), in which case one may expect the following conjecture to be true.

CONJECTURE 2.3. Let a disc F lying in the (x_1, x_2)-plane and centred at the origin be the free surface of a water domain W such that its bottom B is the graph of a rotationally invariant, negative C^2-function given on F. If B forms a non-zero angle with F, then any eigenfunction corresponding to the fundamental eigenvalue attains its minimum and maximum values on ∂F.

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