Modifications to the interfacial wave field moving over variable bottom topography in three dimensions

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Internal waves are generated many places in the world's oceans. The waves induce rather strong currents that have implications to offshore installations. It is particularly the loads and motions induced by the currents that represent the concern, such as, vortex induced vibrations of long cables, pipelines, moorings or risers. Other concerns relate to erosion problems in connection to installations. A fundamental problem relates to the energy cascade taking place when internal waves move over variable bottom bathymetry, particularly the nonlinear part of that motion. If the bottom is very rough, the propagation is both slowed down and damped. Recently, Alam and Mei (2007) studied attenuation of long interfacial waves over a randomly rough sea bed, including the effect of weak nonlinearity, with the wavelength comparable to the horizontal scale of the bathymetry. In Chen and Liu (1996) they developed a variant of the KdV equation for interfacial flow assuming the slowly varying random depth had bathymetric length scale much longer than the characteristic wavelength. We here outline a general method for interfacial wave motion in three dimensions: It is fully nonlinear and dispersive (containing all wavelengths). The full representation of nonlinearity is important, since internal waves typically have large relative vertical excursions. The model evaluates the interfacial wave motion and the induced currents (particularly the bottom currents).

For reference, a few classical works on (linear) wave scattering in a single layer fluid include: Howe (1971), Devillard et al., (1988), Nachbin, (1995), Pelinovsky et al. (1998), Belzons et al. (1988), and Mei and Hancock (2003) - the latter weakly nonlinear.

Fully nonlinear formulation

We consider fully nonlinear motion in three dimensions where an interface separates an upper layer of mean thickness h_2 and density ρ_2 from lower layer of mean thickness h_1 and density ρ_1. Coordinates are introduced with $\mathbf{x} = (x_1, x_2)$ being horizontal and y vertical. The level $y = 0$ separates the layers at rest. The upper layer is covered by a ridid lid, while the lower layer is bounded by a variable bottom determined by $y = -h_1 + \beta(\mathbf{x})$. We assume that the flow in each layer may be modelled by potential theory, with ϕ_j $(j = 1, 2)$ as velocity potentials in the layers. The motion of a non-overturning interface I at $y = \eta$ is considered. The potentials evaluated at the interface are introduced by $\phi_{s1}(\mathbf{x}, t) = \phi_1(\mathbf{x}, y = \eta, t)$ and $\phi_{s2}(\mathbf{x}, t) = \phi_2(\mathbf{x}, y = \eta, t)$, i.e. an index s means the value of the quantity at the actual position of the interface. The kinematic and dynamic boundary conditions at the interface gives

$$\eta_t - V = 0, \tag{1}$$

$$(\phi_{s1} - \mu\phi_{s2})_t + g'\eta + n.l.t.2 = 0, \tag{2}$$

where $\mu = \rho_2/\rho_1$, $g' = (1-\mu)g$ and $V = \phi_{1n}\sqrt{1+|\nabla\eta|^2}$ denotes a scaled normal velocity of the interface. Further, $\nabla = (\partial_{x_1}, \partial_{x_2})$ denotes the horizontal gradient. The set of terms $n.l.t.2$ is derived in Grue (2002, section 6).

The Laplace equation is solved in each layer using a Green function formulation. The resulting integral equations are inverted using Fourier transform, giving (Grue, 2002)

$$\mathcal{F}(V) = -k\tanh kh_2 \mathcal{F}(\phi_{s2}) + n.l.t.a, \tag{3}$$

where $\mathcal{F}$ means Fourier transform, $\mathbf{k}$ wavenumber in spectral space and $k = |\mathbf{k}|$. For the lower layer we get (see Fructus and Grue, 2007):

$$\mathcal{F}(V) = k\tanh kh_1 \mathcal{F}(\phi_{s1}) + \mathrm{i}\mathbf{k}\cdot\mathcal{F}(\beta\nabla\phi_{b1}) + n.l.t.b, \tag{4}$$

$$\mathcal{F}(\phi_{b1}) = \mathcal{F}(\phi_{s1})/\cosh kh_1 - (\tanh kh_1/k)\mathrm{i}\mathbf{k}\cdot\mathcal{F}(\beta\nabla\phi_{b1}) + n.l.t.c, \tag{5}$$

where ϕ_{b1} denotes the wave potential at the bottom boundary, and $n.l.t.b.$, $n.l.t.c.$ denote nonlinear terms.

Effect of a random bottom

The bottom profile is given by $y = -h_1 + \beta(\mathbf{x})$, and we shall assume that the variation of β has zero mean, standard deviation of σ and correlation function given by $<\beta(\mathbf{x})\beta(\mathbf{x}_1)>= \sigma^2(\mathbf{x})\gamma(\mathbf{x}-\mathbf{x}_1)$ where $<>$ represents an average. For the quantity $<\mathcal{F}(\beta)(\mathbf{k})\mathcal{F}(\beta(\mathbf{l})>$ this means

$$\begin{aligned}&<\mathcal{F}(\beta)(\mathbf{k})\mathcal{F}(\beta)(\mathbf{l})>= \iint_{-\infty}^{\infty} d\mathbf{x}_1 e^{\mathrm{i}(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_1} \iint_{-\infty}^{\infty} d\mathbf{x}_1 e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_1)} <\beta(\mathbf{x})\beta(\mathbf{x}_1)> \\ &= 4\pi^2\sigma^2\mathcal{F}(\gamma)(\mathbf{k})\delta(\mathbf{k}+\mathbf{l}),\end{aligned} \tag{6}$$

where δ denotes the Dirac delta-function in two dimensions. The aim is to evaluate the effect of the random bottom on the averaged wave field. From the equations outlined above we obtain for the motion in the lower layer:

Leading approximation:

$$\mathcal{F}(V^0)(\mathbf{k}) = k\tanh kh_1\mathcal{F}(\phi_s)(\mathbf{k}) + n.l.terms, \tag{7}$$

$$\mathcal{F}(\phi_{1b}^0)(\mathbf{k}) = \mathrm{sech}kh_1\mathcal{F}(\phi_s)(\mathbf{k}) + n.l.terms. \tag{8}$$

The effect of a random bottom appears as follows

$$\mathcal{F}(V^r)(\mathbf{k}) = \mathrm{sech}kh_1\mathrm{i}\mathbf{k}\cdot\mathcal{F}(\beta\nabla\mathcal{F}^{-1}(\mathcal{F}({\phi_s}^0)(\mathbf{k})\mathrm{sech}kh_1)) + n.l.terms, \tag{9}$$

$$\mathcal{F}(\phi_{1b}^r)(\mathbf{k}) = -\tanh kh_1\mathrm{i}\mathbf{k}\cdot\mathcal{F}(\beta\nabla\mathcal{F}^{-1}(\mathcal{F}(\phi_s)(\mathbf{k})\mathrm{sech}kh_1)) + n.l.terms. \tag{10}$$

The effect of a random bottom on the averaged motion appears as follows

$$\begin{aligned}&\mathcal{F}(V^{rough})(\mathbf{k}) = -\sigma^2 M(\mathbf{k}h_1)\mathcal{F}(\phi_s)(\mathbf{k}), \\ &M(\mathbf{k}h_1) = \frac{\mathrm{sech}^2 kh_1}{4\pi^2}\iint_{-\infty}^{\infty} d\mathbf{u}\mathcal{F}(\gamma)(\mathbf{u})(\mathbf{u}\cdot\mathbf{k}+k^2)^2\frac{\tanh|\mathbf{u}+\mathbf{k}|h_1}{|\mathbf{u}+\mathbf{k}|},\end{aligned} \tag{11}$$

where the average <> has been used.

Correlation functions are possible in three and two dimensions, i.e.

$$\gamma(x_1, x_2) = e^{-|\mathbf{x}|^2/(2\lambda^2)}, \ \ \gamma(x_1) = e^{-x_1^2/(2\lambda^2)}. \tag{12}$$

Corresponding Fourier transforms read

$$\mathcal{F}(\gamma)(\mathbf{k}) = 2\pi\lambda^2 e^{-k^2\lambda^2/2}, \ \ \mathcal{F}(\gamma)(k_1) = \sqrt{2\pi}\lambda e^{-k_1^2\lambda^2/2}. \tag{13}$$

Dispersion relation

For linear motion with period ω and wavenumber $\mathbf{k}$ (and $k = |\mathbf{k}|$) the dispersion relation is obtained from the prognostic equations (linear)

$$\omega^2 = \frac{g'[k \tanh kh_1 - \sigma^2 M(\mathbf{k}h_1)]}{1 + \mu[\tanh kh_1 - \sigma^2 M(\mathbf{k}h_1)/k]/\tanh kh_2}, \tag{14}$$

which shows that, for fixed ω the wavenumber increases, from plane bottom to rough bottom. This means the wave speed reduces due to a rough sea bottom. More results will be shown at the Workshop.

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