Comments on no definitive trend for the amplitude of the transverse waves generated by a moving body in a two-layer fluid of finite depth

Wei Gang$^{1, 2}$ Su Xiao-bing$^1$ Zhao Xian-qi$^1$

$^1$Faculty of Science, University of Science and Technology, Nanjing, China, gwei@sjtu.edu.cn.
$^2$Shanghai Institute of Applied Mathematics and Mechanics, Shanghai, China, xbsu@163.com

1 Introduction

It is a commonly accepted mechanism that patterns and amplitudes of both surface and interfacial waves generated by a submerged moving body in a two-layer fluid depend on two different modes, a surface-wave mode and an internal-wave mode. In general, the surface-wave mode dominates the surface waves while the internal-wave mode dominates the interfacial waves, so it is allowed to treat the surface waves separately from the internal waves as were done by many authors.

Yeung & Nguyen (1999) first derived out the critical Froude Number $F_{r_c}$, which plays an important role in determining the behavior of the wave pattern, such as its shape and amplitude, and gave an excellent analysis on the ‘cross-coupling influence’ at the free surface or at the interface, i.e. the surface waves associated with the internal-wave mode and the interface-wave mode. A condition that effects of the two modes on the surface or interfacial waves become comparably important was given, which was satisfied for the case of the density ratio $\rho_1/\rho_2$ not close to one, the source approaching to the interface and the ratio of the depth Froude number to the critical Froude Number $F_{r/F_{r_c}}$ approaching to unity.

An noticeable behavior that the coupling term $A_{1, 2}^{(2)}$, i.e. the amplitude of the internal transverse wave due to the surface-wave mode, undergoes an non-monotonic variation with $F_{r/F_{r_c}}$ approaching to unity from below, as shown in figure 17(b) on pp99, was considered ‘more unpredictable’ and ‘no definitive trend’ by Yeung & Nguyen (1999). This opinion makes the authenticity of the linear ship-wave theory under suspicion. In fact, the characteristic parameters given in the figure didn’t satisfy the condition for the ‘cross-coupling influence’ to be considered, so that the internal-wave mode still dominates the internal waves while the effect of the surface-wave mode on the internal waves can almost be omitted. Unfortunately, for the case of a moving source in the upper layer of a two-layer fluid with finite depth, the non-monotonic variation of $A_{1, 2}^{(2)}$ with $F_{r/F_{r_c}}$ can’t be confirmed in some convictive ways, for instance, in a laboratory experiment.

The present study extends Yeung and Nguyen’s theory to the motion of a dipole in the lower layer of a two-layer fluid with finite depth in order to make the comparability between the linear ship-wave theory and Laboratory experiments. An experiment with a set of the same similarity parameters as the theoretical analysis is conducted in a two-layer fluid system composed of kerosene and saline water in a towing-tank. The agreement between present theoretical and experimental results shows that the behavior of the non-monotonic variation of $A_{1, 2}^{(2)}$ with $F_{r/F_{r_c}}$ is intrinsic in the system and can be predicted by the present linear ship-wave theory.

2 Theoretical analysis

By the method of superposing Green’s functions of sources (or sinks), the waves generated a moving dipole in the lower layer of a two-fluid fluid with finite depth have been investigated, which has given in detail by Wei et al (2005).

By calculating the contour integration and using the method of stationary phase (H. Lamb, 1932, Moltgin & Kuznetsov, 1997), we can derive the interfacial elevation at the free surface for the large parameter $r_f/h >> 1$
\[ \zeta^{(2)}(r, \psi) = \sum_{n=1}^{2} \sum_{l=1}^{2} \bar{A}^{(2)}_{nl}(r, \psi) \cos \left( \frac{2 \pi}{h} f_n(\theta, \psi) + \left(1 - \frac{1}{4}\right) \pi \right) \]  

where

\[ \bar{A}^{(2)}_{nl}(r, \psi) = \frac{U^2 R^3 k \cos \theta}{4(1 - \gamma) g A^{(2)}} \left( F^{(2)} - \gamma F^{(1)} \right) \left( \frac{2\pi}{f_n(\theta, \psi)} \right) \left( -\cos \psi \right) \]  

Its non-dimensional form can be written as

\[ A^{(2)}_{nl}(\psi) = \frac{gh^2}{U} |\bar{A}^{(2)}_{nl}| \left[ \frac{8j + c_0 + (-1)^{1/4}}{1 - \gamma} \right]^{1/2} \]  

where \( c_0 = 0 \) for \( \bar{A}^{(2)}_{nl} > 0 \), and \( c_0 = 4 \) for \( \bar{A}^{(2)}_{nl} < 0 \).

Let the ordinate to be the non-dimensional interfacial amplitudes \( A^{(2)}_{nl} \) and the abscissa to be the half angle \( \psi/\psi_0 \) of the Kelvin-ship wave, where \( \psi_0 \) is the maximum half angle in the wake. A suite of characteristic parameters of \( \gamma = 0.82 \), \( h_0/h = 0.286 \), \( \zeta_0/h = -0.429 \) and \( R/h = 0.1785 \), so \( Fr_r = 0.981 \), \( Fr_r = 0.195 \) is taken to obtain effects of variations of \( Fr/F_r \) on \( A^{(2)}_{nl} \) along wave creaselines. The numerical analysis shows that the effect of two wave modes on the interfacial elevation is mainly contributed by the internal one, i.e., \( A^{(2)}_{ll} >> A^{(2)}_{nl} \), as shown in figure 1. It is also seen that as the value of \( Fr/F_r \) increases from zero to one, the amplitude of transverse wave \( A^{(2)}_{n1} \) begins to become large and then decreases gradually, while the amplitude of divergent wave \( A^{(2)}_{n2} \) increases and reaches the magnitude of transverse wave and its disturbance region becomes wider. The non-monotonic variation of \( A^{(2)}_{n1} \) with \( Fr/F_r \) approaching to unity from below can also be found for the value of \( A^{(2)}_{n2} \) due to a source moving in the upper layer of a two-layer fluid system in Yeung’s work (1999).

**3 Comparison with experimental results**

In order to validate the aforementioned theoretical analysis, an experimental investigation on interfacial Kelvin-ship waves generated by a small sphere source moving in the lower layer of a two-layer fluid system is conducted in Fluid Mechanics Laboratory in Shanghai Institute of Applied Mathematics and Mechanics.
Experiments were carried out in a towing tank of 600.0cm long, 50.0cm wide and 50.0cm deep. The stable density stratification of a two-layer fluid system was formed by kerosene and fresh water, whose densities are $\rho_1 = 0.82 \text{ gcm}^{-3}$ and $\rho_2 = 1.00 \text{ gcm}^{-3}$ respectively. The depths of both upper and lower layers were $h_1 = 2.0 \text{ cm}$ and $h_2 = 5.0 \text{ cm}$ respectively, thus $h_1/h = 0.286$, $Fr_1 = 0.981$ and $Fr_2 = 0.195$. A plastic sphere with the diameter $R = 2.5 \text{ cm}$, i.e. $R/h = 0.1785$, was located at the distance 3.0cm blow the interface, i.e. $\zeta_0/h = -0.429$, and was towed along a tensioned stainless-steel guide wire at the constant velocity. The range of towing speed was restricted in 0.00~19.01cm$^{-1}$. The sheet-light technique was employed to carry out flow visualization and the amplitude of the Kelvin-ship wave at the interface were recorded by the CCD camera installed at the side-wall of the tank. Figure 4 gives patterns and amplitudes of the shot interfacial waves for the different towing speeds of 9.48cm$^{-1}$, 11.03cm$^{-1}$, 12.59cm$^{-1}$, 14.10cm$^{-1}$, 15.63cm$^{-1}$ and 19.01 cm$^{-1}$, which correspond to $Fr/Fr_1$ of 0.59, 0.68, 0.78, 0.87, 0.97 and 1.18, respectively.

Due to present chosen characteristic parameters of environments and motions of the sphere, the internal-wave mode dominates the present Kelvin-ship waves at the interface. When $Fr/Fr_1$ is small, the transverse wave is found only in Fig.(2a). As $Fr/Fr_1$ rises, the amplitude of the transverse wave begins to increase (in Fig.2b) and reaches to its maximum (in Fig.2c), and then comes to decay (Fig.2d). Thus, the amplitude of the transverse wave undergoes the non-monotonic variation with $Fr/Fr_1$ approaching to unity from below. The further increase of $Fr/Fr_2$ will results in a combination of the increasing divergent wave and the decreasing transverse wave (in Figs.2d, 2e). When $Fr/Fr_2 > 1$, the transverse wave vanishes and there exists only the divergent wave (in Fig.2f). Here to the evidence in Laboratory experiments consistent with the above theoretical results is obtained.

![Figure 2 Effects of the towing speed on the amplitude and the pattern of the Kelvin-ship wave at the interface](image)

**4 Discussion**

By means of Green’s function for single source (or sink), the velocity potential of a dipole moving in the lower layer of a two-layer fluid with finite depth has been derived. The far-field asymptotic characteristics of the wave elevation at the interface have been deduced by applying the method of stationary phase. The calculated result shows that the amplitude of the transverse
wave $A_{n,2}^{(2)}$ undergoes the non-monotonic variation with $Fr/F_{r*}$ approaching to unity from below. The similar effect, which was considered ‘more unpredictable’ and ‘no definitive trend’ by Yeung & Nguyen, was also found for the case of a source moving in the upper layer of a two-layer fluid. Further, a laboratory study on the internal waves generated by a moving sphere in a two-layer fluid is conducted in a towing tank under the same conditions as that in the theoretical approach. The consistency between the present theory and the laboratory study is examined and confirmed. It is convinced us that the behavior of the non-monotonic variation of $A_{n,2}^{(2)}$ with $Fr/F_{r*}$ is intrinsic in the present discussed system and can be predicted by the liner ship-wave theory.

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Reference


