Persistent modes for water waves and a bulge tube in a narrow channel

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Introduction

A long lossless bulge tube interacts with waves in a rectangular lossless channel. For any wave period and bulge speed, there are two persistent modes, both combining a bulge pressure amplitude A_{b1} or A_{b2} with a wave pressure amplitudes A_{w1} or A_{w2}, in phase in mode 1 and out of phase in mode 2. These modes travel at different speeds c_1 and c_2, and either pure mode will continue unchanged along the channel. If they are both present they will beat, with the result that the combined amplitudes $A_w = A_{w1} + A_{w2}$ and $A_b = A_{b1} + A_{b2}$ change with distance; energy is transfered from wave to bulge and vice-versa. The beat between the modes allows us to calculate the capture width in any situation. We give exact solutions for the persistent modes, which satisfy the Laplace equation for the channel with the correct boundary condition at the tube. To achieve this, the tube is modeled as a narrow vertical structure with width varying vertically to correspond to the attenuation of the wave motion with depth.

We look for a combined solution to the wave equation and the bulge equation which will remain the same at all distances x along the tank. This implies no energy exchange between wave and tube, so the wave must do no work on the tube and vice-versa. Therefore the transverse velocity in the water at the interface must be in quadrature with the wave pressure. Equally the transverse velocity of the tube surface must be in quadrature with the bulge pressure. Furthermore, to satisfy the boundary conditions these transverse velocities must match. It follows that the bulge pressure p_b must be exactly in phase or exactly out of phase with the wave pressure p_w. These two cases define the two persistent modes.

In reality the bulge tube would lie along the centre line of the tank. But to simplify the calculation, we split the bulge tube into two, and put half of it against each wall. To allow precise boundary conditions, imagine that each wall is lined with a flexible vertical membrane enclosing a narrow vertical "bulge tube" between the membrane and the rigid tank wall, the width ξ of the tube being ξ_0 at the surface, decreasing exponentially with depth $-z$. The cross sectional area of a small element of the tube $S = \xi dz$ is controlled by a spring constant such that $dS/S = d\xi/\xi = Ddp$, D being the distensibility, considered to be independent of depth, and p being the pressure difference across the membrane. The velocity of the bulge wave in still water would then be $c_b = 1/\sqrt{D\rho}$. (There will be a vertical pressure gradient inside this tube, but we imagine that there is no vertical flow, perhaps because of some magic horizontal cell structure). With such artificial distensible walls, the equations for the water in the channel and the bulge in the tube can be solved exactly. A more realistic cylindrical bulge tube is discussed below.

Equations for the bulge wave

Assume a long deep rectangular channel of uniform cross section and width $w = 2y_m$. Axes are x in the long direction, y transverse with origin along the centre line and z vertical upwards. There is a wave in the channel of period T with $\omega = 2\pi/T$, propagating without attenuation in the x-direction. In the presence of a water wave with pressure p_w at the surface of the bulge tube, the equation for the bulge pressure p_b was derived in reference[1]

$$\frac{\partial^2 p_b}{\partial t^2} = c_b^2 \frac{\partial^2}{\partial x^2}\{p_b + p_w\} \tag{1}$$

In a persistent mode the ratio of pressures <u>at the bulge tube surface</u> remains constant, so let $p_w/p_b = \alpha$. As these pressures must be either exactly in phase or exactly in anti-phase, α is real but may be negative. Then from (1) the bulge velocity becomes

$$c^2 = c_b^2(1 + \alpha) \tag{2}$$

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or equivalently for a wave of fixed frequency, the original wave number $k_b = \omega/c_b$ is changed to k such that

$$k^2 = k_b^2/(1+\alpha) \tag{3}$$

We take α as positive for Mode1, with value α_1, so the corresponding wave number for the bulge in this mode is

$$k_1^2 = k_b^2/(1+\alpha_1) \tag{4}$$

On the other hand in Mode2, α is negative and it is convenient to put $\alpha = -\alpha_2$. Then the wave number for the bulge in Mode2 is

$$k_2^2 = k_b^2/(1-\alpha_2) \tag{5}$$

As explained above, in a persistent mode the water wave and the bulge wave are synchronised at all distances, so the wave number for the bulge and the wave number for the water wave have to be the same, either k_1 or k_2.

The transverse velocity at the surface of the bulge tube can be calculated as follows. For a bulge wave $p_b = A_b cos(\omega t - kx)$ the variation of tube width is obtained from the distensibility

$$d\xi/\xi = Ddp_b = (1/\rho c_b^2)dp_b \tag{6}$$

so the velocity of the membrane <u>towards the wall</u> is

$$v_y = \frac{\omega \xi A_b}{\rho c_b^2} sin(\omega t - kx) \tag{7}$$

In a persistent mode this transverse velocity must be matched, at the walls of the channel, by an equal transverse velocity in the water wave: this requires a mode in the channel with the appropriate transverse structure. To remain synchronised at all x, the bulge wave, modified as above, and the water wave must have the same value of k.

Equations for the water wave

Consider propagating waves with the vector potential ϕ of the general form

$$\phi = exp\{i\omega t + \zeta x + \eta y + k_0 z\} \tag{8}$$

The vertical boundary conditions define k_0 which in deep water satisfies

$$\omega^2 = gk_0 \tag{9}$$

k_0 is real. k_0 is also the wave number for propagation in the x-direction in a deep channel with no bulge tube. We regard ω and k_0 as fixed. Then the Laplace equation requires

$$\zeta^2 + \eta^2 + k_0^2 = 0 \tag{10}$$

Here ζ and η may be complex, partly imaginary and partly real, as long as (10) is satisfied. We are looking for a solution propagating without attenuation in the x-direction, so ζ must be purely imaginary; and we put $\zeta = ik$. Then from (10)

$$k^2 = k_0^2 + \eta^2 \tag{11}$$

In the normal propagating wave with $k = k_0$, $\eta = 0$ and there is no transverse variation of wave pressure across the channel. In this case there is no transverse velocity at the walls and one cannot satisfy the boundary conditions at the bulge tube. In the desired solution, k can be larger than k_0, in which case η is purely real; or k is smaller than k_0, in which case η is purely imaginary. These two possibilities define the two persistent modes. η cannot be partly real and partly imaginary, as in this case (10) cannot be satisfied without some attenuation in the x-direction.

If η is imaginary (Mode1), put $\eta = ik_y$ with k_y real, and (11) becomes

$$k_1^2 = k_0^2 - k_y^2 \tag{12}$$

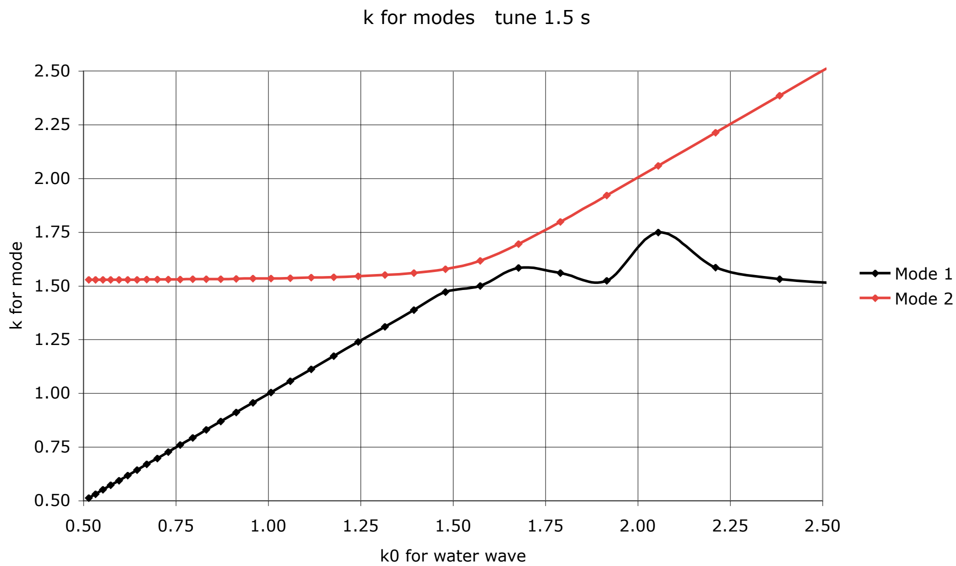

Figure 1: Wave numbers for modes 1 and 2 vs k_0, for $k_b = 1.50$

This shows that k_1 will be smaller than k_0, so the wave will be speeded up.

If on the other hand η is real (Mode2), put $\eta = k_y$ with k_y real, and (11) becomes

$$k_2^2 = k_0^2 + k_y^2 \tag{13}$$

In this case k_2 will be larger than k_0, so the wave will be slowed down.

We need to match the transverse velocity of the water wave at the wall, with the transverse velocity at the surface of the bulge tube. In Mode 1, we look for a version of (8) that is symmetric in y, the most convenient form being

$$\phi = -\frac{A_{w1}}{\rho\omega} exp(k_0 z) sin(\omega t - k_1 x).cos(k_y.y) \tag{14}$$

On the water surface at the walls, with $y = y_m$, this gives wave pressure

$$p_w = -\rho\ \partial\phi/\partial t = A_{w1} cos(\omega t - k_1 x).cos(k_y.y_m) \tag{15}$$

while the transverse outward velocity at the walls is

$$v = \partial\phi/\partial y = \frac{k_y A_{w1}}{\rho\omega} sin(\omega t - k_1 x).sin(k_y.y_m) \tag{16}$$

We see that for small k_y, this outward velocity is positive and in quadrature with the pressure.

Comparing (16) with the velocity at the surface of the bulge tube given by (7), the velocities match if

$$\frac{k_y A_{w1}}{\rho\omega} sin(k_y.y_m) = \frac{\omega \xi_0 A_b}{\rho c_b^2} \tag{17}$$

But the ratio of pressures <u>at the wall</u>, as used in (4) above, is

$$\alpha_1 = p_w/p_b = A_{w1}\ cos(k_y.y_m)/A_b \tag{18}$$

so

$$k_y^2 \alpha_1 \frac{tan(k_y.y_m)}{k_y.y_m} = k_b^2 \frac{\xi_0}{y_m} = k_b^2 \beta \tag{19}$$

where and $\beta = \xi_0/y_m$ is a parameter related to the size of the bulge tube relative to the width of the channel.

We now have three equations (4), (12) and (19), with three unknowns to be determined, α_1, k_1 and k_y, while the given parameters are the wave number for the unperturbed bulge wave k_b, the wave number of the free incident water wave k_0 and the size of the bulge tube specified by β. These equations can be solved numerically by successive approximation and similar equations apply for Mode2 with the hyperbolic function $tanh(k_y.y_m)$ replacing the tangent. For a small bulge tube, tuned to $k_b = 1.50$ with $\beta = 0.001$ in a channel 10 m wide, the unmodified wave number of the water wave was scanned from $k_0 = 0.5 - 2.5$ with the results plotted in Figure 1.

The striking result is that over most of the range of k_0, one persistent mode has a wave number close to k_b, that is it travels at the same speed as the naked bulge, while the other mode travels at the same speed as the undisturbed water wave. The graphs of Mode1 (black line) and Mode2(red line) cannot cross as we see in the figure. So the mode which tracks the water wave changes from Mode1 to Mode2. If the coupling between bulge tube and channel is increased, with larger values of β, then the two lines repel each other and move further apart.

Cylindrical bulge tube

If the bulge tube is cylindrical with cross section S, instead of the assumed narrow vertical wedge, we assume that the transverse volume flow in the water wave should be the same. This gives

$$\beta = \frac{\xi}{y_m} = \frac{\pi S}{\lambda y_m} = \frac{k_0 S}{2 y_m} \tag{20}$$

Capture width

Now that we have the persistent modes, we can calculate how they beat as the waves propagate along the channel and so how power is transferred from water to bulge and vice versa. If the starting condition at $x = 0$ has waves in the channel but no bulge, then Mode1 cancels Mode2 for the bulge pressure. We assume $A_b = 1$ for each mode. As the modes progress along the tank, they will develop a phase difference so the bulge pressure amplitude becomes

$$p_b = cos(k_1.x) - cos(k_2.x) = 2\ sin\{(k_1 - k_2)x/2\}.sin\{(k_1 + k_2)x/2\} \tag{21}$$

Putting x equal to the length of the tube, the bulge power at the end is $P_b = p_b^2 S/2\rho c_b$ where S is the area of the tube, which we take as $S = 2\xi_0/k_0$. This determines the power captured. Comparing with the power in the water wave, gives the capture width.

Further results and calculated wave patterns in the channel will be presented.

References

[1] F.J.M. Farley & R.C.T. Rainey "Anaconda, the bulge wave sea energy converter", 5 Nov 2006; *(download from http://www.bulgewave.com)*