Complete and explicit asymptotics of solutions to the linearized Shallow water equations generated by localized perturbations

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The long waves and large vortices of the small amplitudes in the basin with nonuniform bottom with depth $D(x)$ and the basic flow $V(x,t)$ and the elevation of free surface $R(x,t)$ are described by the linearized Shallow water equations on the β-plane

$$\frac{\partial \eta}{\partial t} + div(c^2 u + \eta V) = 0, \quad \frac{\partial u}{\partial t} + (V, \nabla)u + \frac{\partial V}{\partial x}u + \omega T u + \nabla \eta = 0, x = (x_1, x_2) \in \mathbb{R}^2_x, \quad (1)$$

here the velocity $c^2(x,t) = R(x,t) + D(x)$, $\omega = \omega_0 + \beta x_2$ is the Coriolis frequency on the β-plane, $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We use the dimensionless variables and consider the Cauchy problem with the localized initial data

$$u|_{t=0} = u^0(\frac{x}{\mu}) = \begin{pmatrix} u_1^0(\frac{x}{\mu}) \\ u_2^0(\frac{x}{\mu}) \end{pmatrix}, \quad \eta|_{t=0} = \eta^0(\frac{x}{\mu}), \quad (2)$$

where μ is a small parameter, $u_j^0(z), \eta^0(z)$ are smooth functions decaying faster than $1/|z|^\beta$ as $|z| \to \infty$ with $\beta > 2$. We assume also that the $k-th$ derivatives decay faster than $1/|z|^{\beta+|k|}$. The small parameter μ characterizes the localization of the initial perturbation. We assume that the depth $D(x)$ and the background $V(x,t), R(x,t)$ change slowly. The solutions to problem (1),(2) can be used for description of tsunami waves in the ocean and mesoscale vortices in the atmosphere.

We suggest a new asymptotic representation for the solutions to problem (1),(2). This representation is given in a form of the generalization of the construction known as the Maslov canonical operator and is based on a simple relationship between fast decaying and fast oscillating solutions. The complete solution is decomposed into two parts : the wave part $\begin{pmatrix} u^w \\ \eta^w \end{pmatrix}$ and the vortical part $\begin{pmatrix} u^v \\ \eta^v \end{pmatrix}$. With time the supports of these two vector functions separate: the wave part will be localized in the neighborhood of closed curve γ_t^w which is the wave front on the plane $\mathbb{R}^2_x$, whereas the vortical part will be localize in the neighborhood of the point r_t^v on $\mathbb{R}^2_x$). (See Fig.1)

Consider two Hamiltonian systems

$$\dot{p} = -{}^t V_x - |p|\nabla c, \quad \dot{x} = V\frac{p}{|p|}c, \quad \text{(the Hamiltonian is:} \quad H^w(p,x,t) = \langle V(x), p\rangle \pm |p|c(x,t)),$$

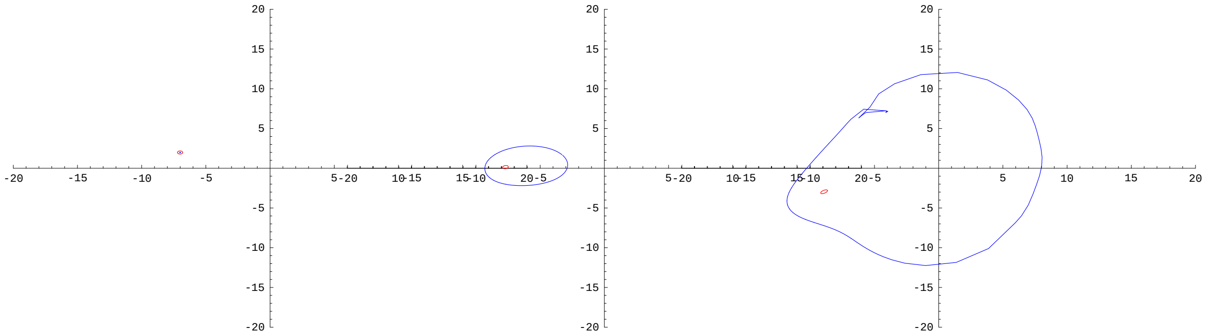

Figure 1: The center of the vortex and the front of the wave for various t, the depth $D(x_1,x_2) = 0.1(1+3\tanh^2(2x_1+x_2+1))/\cosh^2(\frac{\sqrt{4x_1^2+x_2^2}}{5})$, and basic velocity $V(x_1,x_2) = 0.001\big((\cos x_2 + x_1), x_1\big)$. One can see the appearance of the focal points on the front at the third picture.

$$\dot p = -{}^tV_x p, \quad \dot x = V, \qquad (\text{the Hamiltonian is:} \quad H^v(p,x) = \langle V(x), p\rangle),$$

corresponding to the wave and vortical modes respectively. For any fix time t both sets mentioned above are defined as the ends of the x-components $x = X^w(t,\psi), x = X^v(t)$ of their trajectories $(p = P^w(t,\psi), x = X^w(t,\psi), \quad p = P^v(t,\psi), x = X^v(t))$ starting from the circle $\{p = \mathbf{n}^0(\psi) \equiv \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}, x = 0\}$ in the phase space $\mathbb{R}^4_{p,x}$, where $\psi \in [0,2]\pi$ is the angle (a parameter). One obtains the rays fixing ψ and taking x-components of trajectories.

The beautiful fact based on boundary layer ideas is that the general formulas could be simplified and expressed in terms of the trajectories $(p = P^w(t,\psi), x = X^w(t,\psi)$, $p = P^v(t,\psi), x = X^v(t))$ and the Fourier transform $\widetilde{u}^0(k), \widetilde{\eta}^0(k)$ of initial functions u^0, η^0 only.

For the wave mode this statement is true including the case when the focal (turning) and self intersection points appear on the front. Moreover we show that the time $t = 0$ corresponds to the strong focal point of the wave part of the asymptotic solution: the the curve $\{x = X^w(t,\psi)\}$ tightens to the point $x = 0$, and that vortical part corresponds to the strong focal point for any time t. Due to the last fact the vortical part has the same structure as the initial velocity: it is localized in the neighborhood of the point $X^v(t)$ and velocity component u^v of asymptotic solution to (1), (2) has the form

$$u^v = \frac{c^2(X(t),t)}{c^2(0,0)} g\Big(\frac{Z(t)^{-1}(x - X(t))}{\mu}\Big) + O(\mu), \quad \eta^v = O(\mu),$$

where

$$g(y,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{n}_\perp\big((Z^*)^{-1}(t)k\big)\frac{|k|}{|(Z^*)^{-1}(t)k|} < \widetilde{u}^0(k), \mathbf{n}_\perp(k) > e^{i<k,y>}dk, \quad \mathbf{n}(p) = \frac{1}{|p|}\begin{pmatrix}p_2\\ -p_1\end{pmatrix}$$

and $Z(t)$ is the matrix solution (Cauchy matrix) of the equation

$$\dot Z = V_x(X^v(t),t)Z, \quad Z|_{t=0} = E, \quad E \quad \text{is} \quad 2\times 2 \quad \text{unit} \quad \text{matrix}.$$

The explicit formulas for the wave part are different in the neighborhood of the regular arcs of the front where $|\frac{\partial X^v}{\partial\psi}| \neq 0$ and in the neighborhood of the focal points where $|\frac{\partial X^v}{\partial\psi}| = 0$.

Let us describe the asymptotics of wave part in the first situation restricting ourselves to $\eta-$ component and taking $V(x,t)=0, R(x,t)=0, u^0=0$. We define in a neighborhood of the wave front coordinate y, where $|y|$ is the distance between the point x belonging to a neighborhood of the wave front and the wave front. For this aim we will take $y \geqslant 0$ for the external subset of the wave front and $y \leqslant 0$ and for the internal subset. Then a point x of the neighborhood of the wave front is characterized by two coordinates: $\psi(x,t)$ and $y(x,t)$, where $\psi(x,t)$ is defined by the condition that the vector $y = x - X(t,\psi)$ is orthogonal to the vector tangent to the wave front in the point $X(t,\psi)$, so $\langle y, X_\psi(t,\psi)\rangle = 0$. The phase is defined by $S(x,t) = \sqrt{\frac{D(0)}{D(X^w(t,\psi(t,x)))}}y$. When the focal points appear on the wavefront, then some arcs of the front could be very close one to another or even intersect. In this case the elevation $\eta(x,t)$ of the wave in a point x belonging to a neighborhood of these arcs can be presented as a sum of the contributions coming from different $\psi_j(x,t)$, $y_j(x,t)$, and $S_j(x,t)$ with index j. Also important topological characteristics of the focal points appear here. They are the so-called the Maslov indices coinciding here with the Morse indices $m_j = m(\psi_j(x,t),t)$. The index $m_j = m(\psi_j(x,t),t)$ takes one of the following integer values: 0,1,2,3 and it is *the number of focal point staying on a trajectory coming to the point* $x = X^w(t,\psi)$. Finally in the neighborhood of the front but outside of some neighborhood of the focal points the wave field is the sum of the fields

$$\eta(x,t) = \sum_j \{ \frac{\sqrt{\mu}}{\sqrt{|X_{\psi_j}(t,\psi)|}} \sqrt[4]{\frac{D(0)}{D(X(t,\psi))}}\,\mathrm{Re}\,\big[e^{-\frac{i\pi m_j}{2}} F(\frac{S_j(x,t)}{\mu}, \mathbf{n}^0(\psi))\big]\}\big|_{\psi=\psi_j(x,t)} + o(\mu^{3/2}).$$

Here

$$F(z,\psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_0^\infty \sqrt{\rho}\tilde{\eta}^0(\rho,\psi) e^{iz\rho} d\rho.$$

This formula establishes the connection between initial localized perturbations and wave profiles near the wave fronts. Now we derive some consequences from this formula. Since the phase $S(x,t)$ is equal to zero on the wave front and $|S(x,t)|/\mu$ increases rapidly going out from it, then maximum of $|\eta|$ is attained in a neighborhood of the wave front. Moreover, $\eta(x,t)$ can exhibit few oscillations depending on the properties of the function $F(z,\psi)$ (which in turn, depend on the form of the initial disturbance). The second factor in can be interpreted as two dimensional analogue of *the Green law*, well known in the theory of tidal waves in the channels: amplitude of η increases as $1/\sqrt[4]{H(x)}$ when the depth $H(x)$ of the basin decreases; the factor $1/\sqrt{|X_\psi|}$ is connected to *the divergence of the rays*, in other words if a smaller number of rays goes through a neighborhood of the point $X(t,\psi)$, the smaller will be the amplitude of the wave field. The factor $\frac{H(0)}{H(X(t,\psi))}$ in the phase $S(x,t)$ expresses the phenomena known as the "contraction" of the wave profile and explains the fact that the wave length of a tsunami decreases when the wave approaches the coast. The indices m_j change the behavior of the functions determining the wave profile. We illustrate with help of Fig. the the profiles near the wavefront of the waves generated by special disturbance in the basin with the depth presented in Fig. .

In the neighborhood of the focal points it is necessary to use different formulas and the solution is expressed via the integrals

$$g_n(z_1,z_2,\psi) = \int_{-\infty}^\infty d\xi \int_0^\infty \sqrt{\rho} d\rho \sqrt{\rho}\tilde{\eta}^0(\rho\mathbf{n}(\psi)) \exp\{i\rho\big(z_2 - \xi z_1 \pm \frac{\xi^{n+1}}{(n+1)!}\big)\}.$$

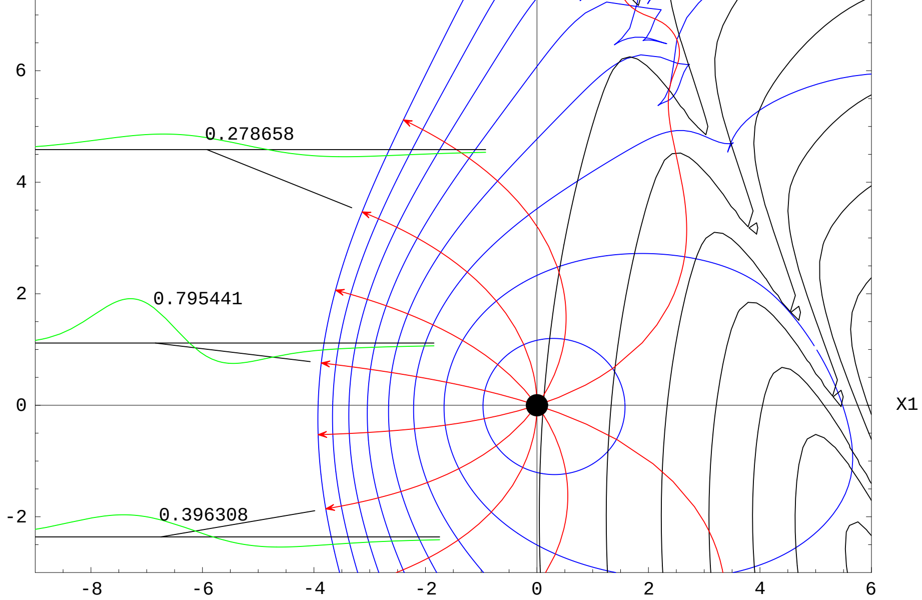

Figure 2: Fronts and profiles

Some results of this research were published in [1],[2],[3].

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