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Time-harmonic ship waves with the effect of surface tension and fluid viscosity

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The potential flow in a viscous fluid generated by a source pulsating sinusoidally and moving with constant horizontal velocity is considered within the framework of linear Oseen equations. The combined effect of fluid viscosity and surface tension on the potential function below the water surface is studied. It is shown that the wave form of oscillations is largely modified by the surface tension while the wave amplitude of most short waves is attenuated by the fluid viscosity. Unlike pure-gravity waves, the non-physical properties of high oscillations and strong singularities revealed in Chen & Wu (2001) disappear in the fundamental solution of time-harmonic ship waves with the effect of surface tension and fluid viscosity.

1. Time-harmonic Oseen equations

We consider the lower half-space filled with water limited on the top by the water-air interface. A Cartesian coordinate system is defined by placing the (x, y)-plane coincided with the undisturbed free surface and the z-axis oriented positively upward. This reference system is moving at the mean forward speed U along the positive x-direction same as that of the source which is pulsating harmonically at a frequency denoted by ω.

In this gravity-dominant fluid domain, the reference length L, the acceleration of gravity g and the water density ρ are used to define the nondimensional coordinates (x, y, z), the time t, the fluid velocity $\mathbf{u} = (u, v, w)$, the velocity potential Φ and the dynamic pressure P with respect to $(L, \sqrt{g/L}, \sqrt{gL}, \sqrt{gL^3}, \rho g L)$, respectively, while the nondimensional frequency f, the Froude number F and the Brard number τ by

$$(f, F, \tau) = (\omega\sqrt{L/g}, U/\sqrt{gL}, U\omega/g) \tag{1}$$

respectively. Under the hypothesis of linear and time-harmonic motions, all physical quantities are expressed by the form $Q(\mathbf{x}, t) = \Re\{q(\mathbf{x}) \exp(-ift)\}$ with $i = \sqrt{-1}$ and t the nondimensional time, so that only their complex associate $q(\mathbf{x})$ is involved in the following development.

We study the flow in the fluid domain $\mathbf{x} = (a, b, c \leq 0)$ due to a point source of unit strength located at $\mathbf{x}' = (a', b', c' \leq 0)$. By assuming the incompressibility, the fluid flow is governed by the continuity equation :

$$\nabla \cdot \mathbf{u} = \delta(\mathbf{x}-\mathbf{x}') \tag{2a}$$

and the momentum equation :

$$(F\partial_x - if)\mathbf{u} = -\nabla P + \epsilon \nabla^2 \mathbf{u} \tag{2b}$$

where $\epsilon = \mu/(\rho\sqrt{gL^3})$ with μ the fluid viscosity and $\delta(\cdot)$ is the Dirac delta function.

On the free surface $z = \eta(x, y, t)$, the boundary conditions are linearized by assuming small wave amplitudes and written on the undisturbed free surface $c = 0$:

$$(F\partial_x - if)\eta = w \tag{3a}$$

as the kinematic condition stating no fluid particles cross the free surface and

$$\epsilon(u_z + w_x) = 0 = \epsilon(v_z + w_y) \tag{3b}$$

$$\eta - \sigma(\eta_{xx} + \eta_{yy}) + 2\epsilon w_z = P \tag{3c}$$

as the dynamic conditions representing the vanishing of shear stress in both x and y directions (3b) and the equation of normal stress (3c). In (3c), $\sigma = T/(\rho g L^2)$ with T is the surface tension of water-air interface.

2. Integral solutions by Fourier transform

To solve the boundary-value problem preceding defined (2-3), the unknowns $(\mathbf{u}, P)$ are decomposed as the sum of an *unbounded* singular Oseen flow $(\mathbf{u}^S, P^S)$ and the regular flow $(\mathbf{u}^R, P^R)$ which represents the free-surface effect. Thus, we write :

$$(\mathbf{u}, P) = (\mathbf{u}^S, P^S) + (\mathbf{u}^R, P^R) \tag{4}$$

The singular Oseen flow $(\Phi^S, \mathbf{u}^S)$ satisfies

$$\nabla \cdot \mathbf{u}^S = \delta(\mathbf{x}-\mathbf{x}') \tag{5a}$$

$$(F\partial_x - if)\mathbf{u}^S - \epsilon\nabla^2\mathbf{u}^S = -\nabla P^S \tag{5b}$$

which gives the solution :

$$\mathbf{u}^S = \nabla\Phi^S \quad \text{and} \quad P^S = -(F\partial_x - if)\Phi^S \tag{6a}$$

with :

$$\Phi^S = -1/(4\pi|\mathbf{x}-\mathbf{x}'|) \tag{6b}$$

Using the integral representation of Φ^S, we have $\mathbf{u}^S$ written as :

$$\mathbf{u}^S = \frac{-1}{8\pi^2}\,\nabla\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta\, e^{-k|c-c'|+i(\alpha x+\beta y)}/k \tag{6c}$$

with $(x, y) = (a-a', b-b')$ and $k = \sqrt{\alpha^2+\beta^2}$. Furthermore, we write the regular Oseen flow $(\mathbf{u}^R, P^R)$ as the sum of an irrotational and a solenoidal vectors :

$$\mathbf{u}^R = \nabla\Phi + \mathbf{u}^T \tag{7}$$

where the scalar function $\Phi(\mathbf{x}, t)$ represents the irrotational flow while $\mathbf{u}^T$ the rotational flow. The Oseen flow $(\Phi, \mathbf{u}^T)$ taking account of free-surface effect satisfies :

$$\nabla^2\Phi = 0 = \nabla\cdot\mathbf{u}^T \tag{8a}$$

$$(F\partial_x - if)\mathbf{u}^T = \epsilon\nabla^2\mathbf{u}^T \tag{8b}$$

The dynamic pressure P^R is defined by :

$$P^R = -(F\partial_x - if)\Phi \tag{9}$$

The boundary conditions (3) can now be expressed in terms of $(\mathbf{u}^S, P^S, \Phi, \mathbf{u}^T)$ on the undisturbed free surface $(c=0)$:

$$(F\partial_x - if)\eta - (\Phi_z + w^T) = w^S \tag{10a}$$

$$2\Phi_{zx} + u_z^T + w_x^T = -(u_z^S + w_x^S) \tag{10b}$$

$$2\Phi_{zy} + v_z^T + w_y^T = -(v_z^S + w_y^S) \tag{10c}$$

$$(F\partial_x - if)\Phi + \eta - \sigma(\eta_{xx} + \eta_{yy}) + 2\epsilon(\Phi_{zz} + w_z^T) = -(F\partial_x - if)\Phi^S - 2\epsilon w_z^S \tag{10d}$$

Now we introduce the Fourier integral transform for $(\eta, \Phi, \mathbf{u}^T)$ as :

$$\begin{pmatrix}\eta\\ \Phi\\ \mathbf{u}^T\end{pmatrix} = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \begin{pmatrix}\tilde{\eta}\\ \tilde{\Phi}_0 e^{kc}\\ \tilde{\mathbf{u}}_0^T e^{k_\epsilon c}\end{pmatrix} e^{i(\alpha a+\beta b)} \tag{11}$$

in which, we have used the notations :

$$k_\epsilon = \sqrt{k^2 + i(F\alpha - f)/\epsilon}$$

$$\tilde{\Phi}_0 = \tilde{\Phi}(a, b, c=0)$$

$$\tilde{\mathbf{u}}_0^T = \tilde{\mathbf{u}}^T(a, b, c=0)$$

Introducing the Fourier integral transform (11) on the left hand side of (10) as well as $\nabla\cdot\mathbf{u}^T=0$ (8a), and the integral form of $(\Phi^S, \mathbf{u}^S)$ on the right hand side of (10), a system of linear equations is obtained by equating the integrands on both sides of (10) and (8a), for the five unknowns :

$$\begin{pmatrix} i(F\alpha-f) & -k & 0 & 0 & -1\\ 0 & 2ik\alpha & k_\epsilon & 0 & i\alpha\\ 0 & 2ik\beta & 0 & k_\epsilon & i\beta\\ 1+\sigma k^2 & i(F\alpha-f)+2\epsilon k^2 & 0 & 0 & 2\epsilon k_\epsilon\\ 0 & 0 & i\alpha & i\beta & k_\epsilon \end{pmatrix} \begin{pmatrix}\tilde{\eta}\\ \tilde{\Phi}_0\\ \tilde{u}_0^T\\ \tilde{v}_0^T\\ \tilde{w}_0^T\end{pmatrix} = \begin{pmatrix}1\\ -2i\alpha\\ -2i\beta\\ i(F\alpha-f)/k+2\epsilon k\\ 0\end{pmatrix} \frac{e^{kc'-i(\alpha a'+\beta b')}}{8\pi^2} \tag{12}$$

which gives :

$$\tilde{\Phi}_0 = (1/k + 2\mathcal{N}/\mathcal{D})e^{kc'-i(\alpha a'+\beta b')}/(8\pi^2) \tag{13}$$

with

$$\mathcal{N} = 1 + \sigma k^2 - 4\epsilon^2 k^2 k_\epsilon \tag{14a}$$

$$\mathcal{D} = -[i(F\alpha - f) + 2\epsilon k^2]^2 - k - \sigma k^3 + 4\epsilon^2 k^3 k_\epsilon \tag{14b}$$

Similar results for $(\tilde{\eta}_0, \tilde{u}_0^T, \tilde{v}_0^T, \tilde{w}_0^T)$ with the same function $\mathcal{D}$ in denominator can be obtained. The wave elevation η has been considered in a number of studies as in Lu & Chwang (2005). Since the amplitude functions of $(\tilde{u}_0^T, \tilde{v}_0^T, \tilde{w}_0^T)$ are of order $(\epsilon^{1/2}, \epsilon^{1/2}, \epsilon)$, respectively, we are interested here to the potential function Φ which is obtained by introducing (13) back to the integral transform (11) :

$$\Phi = 1/(4\pi|\mathbf{x}-\mathbf{x}_1'|) + \Phi^F \tag{15}$$

with $\mathbf{x}_1' = (a', b', -c')$ is the mirror point of $\mathbf{x}' = (a', b', c')$ with respect to the mean free surface $z = 0$, and Φ^F is the wave potential function :

$$\Phi^F = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, \frac{\mathcal{N}}{\mathcal{D}} \, e^{kz+i(\alpha x+\beta y)} \tag{16}$$

in which $(x, y, z) = (a - a',\, b - b',\, c + c')$ while $\mathcal{N}$ and $\mathcal{D}$ are given in (14). The potential function Φ^F representing waves due to free-surface effect is analyzed below.

3. Potential function of time-harmonic ship waves

The potential function defined by the Fourier representation (16) involves $\mathcal{N}e^{kz}$ and $\mathcal{D}$ which are often called as the amplitude and dispersion functions, respectively. Since the fluid viscosity $\epsilon \ll 1$, the expression (16) can be rewritten as

$$4\pi^2 F^2 \Phi^F = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, \frac{1+\sigma k^2}{D} \, e^{kz+i(\alpha x+\beta y)} \tag{17a}$$

by neglecting the terms of order $O(\epsilon^{3/2})$ or higher. The dispersion function D is then given by :

$$D = (k\cos\theta - \tau)^2 - k - \sigma k^3 - 4i\epsilon(k\cos\theta - \tau)k^2 \tag{17b}$$

as a complex function of (k, θ) with $(\alpha, \beta) = k(\cos\theta, \sin\theta)$. In obtaining (17a) and (17b), we have used the scale factor $(F^2, 1/F^2)$ for the coordinates (x, y, z) and Fourier variables (α, β, k), respectively. Thus the parameters (σ, ϵ) are redefined as :

$$\sigma = T/(\rho g L^2 F^4)\ ; \qquad \epsilon = \mu/(\rho\sqrt{gL^3}F^4)$$

accordingly.

In the limit of $\epsilon \to 0$, Noblesse & Chen (1995) has performed the analysis of the potential function like (17a) without the effect of surface tension ($\sigma = 0$). The time-harmonic potential expressed by the double Fourier integral is decomposed into a wave component which propagates into the far field, and a local component which is only significant in the near field. It is shown that the wave component can be obtained by performing an integration over a strip in the Fourier plane cross the dispersion curves defined by $D = 0$. Following the same principle, we start to find the complex solutions of :

$$k(\theta) = k^R(\theta) + ik^I(\theta) \quad \text{to satisfy} \quad D(k, \theta) = 0 \tag{18}$$

for a real θ and define the dispersion curves $\mathcal{L}$ in the real Fourier plane $(\alpha, \beta) = k^R(\cos\theta, \sin\theta)$. The double Fourier integral (17a) is now performed along the dispersion curve and in its normal direction by changing the integral variables. The integration in the direction normal to the dispersion curve is analytically evaluated by taking its leading order and the resultant integral is written along the dispersion curves $\mathcal{L} \subset k^R(\theta)$ by :

$$\Phi^W = i2\pi\int_{\mathcal{L}} ds \; e^{kz+i(\alpha x+\beta y)}(\Sigma_1+\Sigma_2)\mathcal{A}\, e^{-(\alpha^I x+\beta^I y)\Sigma_1}/|\nabla D^R| \tag{19}$$

in which we have used :

$$(\alpha^I, \beta^I) = |k^I|(\cos\theta, \sin\theta),\ \Sigma_1 = \mathrm{sign}(k^I),\ \Sigma_2 = \mathrm{sign}(\alpha x+\beta y),\ \mathcal{A} = k(1+\sigma k^2)D_k^R/(k^R D_k)$$

with $D^R = \Re\{D(k,\theta)\}$ and $|\nabla D^R| = \sqrt{(\partial D^R/\partial\alpha)^2 + (\partial D^R/\partial\beta)^2}$ as well as $(D_k^R, D_k) = (\partial D^R/\partial k, \partial D/\partial k)$.

The wave component defined by (19) is supposed to be wavy component in the far field and the difference

$$4\pi^2 F^2 \Phi^F - \Phi^W \to 0 \quad \text{smoothly as} \quad \sqrt{x^2+y^2} \to \infty \tag{20}$$

Following the stationary-phase method used in Chen & Noblesse (1997), the analysis of the wave-component line integral (19) gives the direct relationship between the dispersion curves in the Fourier plane and the corresponding wave systems on the free surface. The dispersion curves defined by $k^R(\theta)$ are symmetrical with respect to $\beta = 0$ and there exist three or two distinct dispersion curves for $\tau < 1/4$ or $\tau > 1/4$, respectively. The dispersion curves are nearly identical as those in Chen (2005) in the zone $\sigma^{1/3}k \ll 1$. However, the most striking difference is that the two *open* dispersion curves on the left and right half parts of the Fourier plane are now *closed*. This closure of dispersion curves bears two major significations. The wavenumbers are finite not infinite as for pure gravity waves and the divergent waves including in the inner-V, outer-V and ring-fan wave systems are largely modified according to the relationship between the dispersion relation and far-field waves established in Chen & Noblesse (1997).

Furthermore, the factor $e^{-(\alpha^I x + \beta^I y)\Sigma_1}$ in (19) associated with the fluid viscosity with the argument $(\alpha^I x + \beta^I y)\Sigma_1$ always positive for $\Sigma_1 + \Sigma_2 \neq 0$, represents an exponential decay for large values of k which shows that waves of large wavenumbers are rapidly attenuated.

4. Discussions and conclusions

By ignoring the effect of surface tension and fluid viscosity, the time-harmonic potential function due to a pulsating and advancing source was analyzed in Chen & Wu (2001) by using the wave-component integral along the open dispersion curves. The asymptotic calculation of the integral along the portion of the open dispersion curves at large values of wavenumber yields an analytical expression which reveals the peculiar behavior of highly oscillations with singular amplitudes when a field point approaches to the track of the source point at the free surface. These gravity waves of very short wavelength cause substantial difficulties in the numerical prediction of the motion of a ship advancing in waves.

The present study by including the effect of both surface tension and fluid viscosity shows that the singular and highly oscillatory properties of pure-gravity waves is manifestly non-physical and that the effect of surface tension modifies the wave forms and fluid viscosity attenuates most short waves. These benefits will be much more enjoyed in the numerical development of practical computation method.

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