

Hydrodynamic diffraction by multiple elliptical cylinders

by

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1 General

The present paper treats the hydrodynamic diffraction problem between multiple bottom seated elliptical cylinders and regular surface wave trains. Linear potential theory is employed and the solution method is based on the semi-analytical formulation of the incident and the diffracted surface waves' potentials around the elliptical bodies. Virtually, the mathematical process has many similarities with the well known analytical technique which is implemented for the hydrodynamic interaction between arrays of circular cylinders and regular wave train. Nevertheless, the problem associated with arrays of elliptical cylinders is admittedly more complicated as it involves the ill-familiar periodic and radial Mathieu functions and requires the implementation of an addition theorem, similar in concept with the Graf's addition theorem for Bessel functions, but relevant to the periodic and radial Mathieu functions.

2 Mathematical formulation and solution

The arrangement of the elliptical cylinders depicted in Fig. 1 is investigated. All bodies are considered fixed on the bottom and exposed to the action of monochromatic incident waves of frequency ω and linear amplitude $H/2$, propagating at angle α to the positive x direction. The bodies are fixed in water of depth h . The large and the small radii of the k th body are denoted by a_k and b_k respectively. Elliptical cylindrical coordinates (u, v, z) are employed, $u=\text{constant}$, $v=\text{constant}$ being orthogonally intersecting families of confocal ellipses and hyperbolae, respectively. The z -axis is fixed on the bottom, pointing vertically upwards. It is assumed that the reader is familiar with Mathieu functions and their definitions and properties [1-5]. The velocity potentials must satisfy the Laplace equation and the conditions on the bottom and the free surface. The total velocity potential must also satisfy the kinematical condition on the wetted area of the bodies

$$\left(\frac{\partial \varphi}{\partial u} \right)_{u=u_0} = 0, \quad 0 \leq z \leq h \quad (1)$$

where u_0 stands for the radial boundary of any body with respect to its local elliptical coordinate system. In addition, the diffraction potentials must comply with the appropriate radiation condition at infinity, which in elliptical coordinates is expressed as

$$\lim_{u \rightarrow \infty} (c \cosh u)^{1/2} \left\{ \frac{1}{c \sinh u} \frac{\partial \varphi^{(D)}}{\partial u} - ik_0 \varphi^{(D)} \right\} = 0 \quad (2)$$

where k_0 is the wave number and $c=(a^2-b^2)^{1/2}=a\varepsilon$ with ε being the elliptic eccentricity given by $\varepsilon^2=1-(b/a)^2$.

The potentials shown in the sequel are given in nondimensional form with normalization parameter- $i\omega(H/2)h$. The incident wave potential, expressed with respect to the local elliptical coordinate system of body k of the arrangement, is given by

$$\varphi^{(I)} = \frac{2g}{\omega^2 h} \frac{\cosh(k_0 z)}{\cosh(k_0 h)} \Lambda_k \left\{ \sum_{m=0}^{\infty} i^m \text{Mc}_m^{(1)}(u_k; q_k) \text{ce}_m(v_k; q_k) \text{ce}_m(\alpha; q_k) + \sum_{m=1}^{\infty} i^m \text{Ms}_m^{(1)}(u_k; q_k) \text{se}_m(v_k; q_k) \text{se}_m(\alpha; q_k) \right\} \quad (3)$$

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where g is the acceleration due to gravity, $\text{Mc}_m^{(1)}$, $\text{Ms}_m^{(1)}$ are the even and odd radial Mathieu functions of the first kind and ce_m and se_m are the even and odd periodic Mathieu functions. The index m denotes the order and $q_k = (k_0 c_k / 2)^2$ is the Mathieu parameter. Finally, $\Lambda_k = e^{ik_0(X_k \cos \alpha + Y_k \sin \alpha)}$ where X_k and Y_k are the Cartesian coordinates of the center of body k with respect to the global Cartesian coordinate system.

The total diffraction potential should involve the diffraction potentials due to the wave scattering caused by all bodies of the arrangement. As a result, it holds that

$$\begin{aligned} \varphi^{(D)} = \sum_{k=1}^N \varphi_k^{(D)} = \cosh(k_0 z) \sum_{k=1}^N \left\{ \sum_{m=0}^{\infty} i^m \tilde{A}_m^{(k)} \text{Kc}_m^{(k)} \text{Mc}_m^{(3)}(u_k; q_k) \text{ce}_m(v_k; q_k) \right. \\ \left. + \sum_{m=1}^{\infty} i^m \tilde{B}_m^{(k)} \text{Ks}_m^{(k)} \text{Ms}_m^{(3)}(u_k; q_k) \text{se}_m(v_k; q_k) \right\} \end{aligned} \quad (4)$$

where N is the number of bodies, $\text{Mc}_m^{(3)}$, $\text{Ms}_m^{(3)}$ are the even and odd radial Mathieu functions of the third kind, while $\tilde{A}_m^{(k)}$ and $\tilde{B}_m^{(k)}$ are the unknown Fourier coefficients. Finally, $\text{Kc}_m^{(k)} = \text{Mc}_m^{(1)'}(u_{k_0}; q_k) / \text{Mc}_m^{(3)'}(u_{k_0}; q_k)$, $\text{Ks}_m^{(k)} = \text{Ms}_m^{(1)'}(u_{k_0}; q_k) / \text{Ms}_m^{(3)'}(u_{k_0}; q_k)$ are assisting parameters which were included artificially. Fourier coefficients are calculated using the kinematical condition on the bodies given by Eq. (1). It follows that Eq. (4) should be expressed with respect to the local elliptical coordinate system of an arbitrarily selected body k . Thus, an appropriate addition theorem for Mathieu functions is required.

3 Addition theorem for Mathieu functions

The existence of an addition theorem for Mathieu functions was shown by Særmærk [5] who extended the formulas reported in Meixner and Schäfke [3] in terms of the Bessel functions. The addition theorem introduced by Særmærk [5] was expressed in terms of the radial and periodic Mathieu functions in the notation of [3]. Unfortunately, there are several notations used in the literature for expressing Mathieu functions and by general confection, the notation that has prevailed is that of Abramowitz and Stegun [1], which is also employed in the present. Thus, for the purposes of the present contribution, the addition theorem had to be transformed in order to involve the even and odd periodic and radial Mathieu functions. Under these conditions, the addition theorem should be split to different expressions for even and odd Mathieu functions. After extensive mathematical manipulations it can be shown that the associated theorems can be written as

$$\begin{aligned} & \text{Mc}_m^{(3)}(u_j; q_j) \text{ce}_m(v_j; q_j) \\ &= \sum_{q=0}^{\infty} \text{Mc}_q^{(1)}(u_k; q_k) \text{ce}_q(u_k; q_k) \frac{1}{2} \sum_{s,p=-\infty}^{\infty} (-1)^{(p-m+q-s)/2} A_s^q(q_k) A_p^m(q_j) \text{H}_{-p+s}(k_0 R_{jk}) e^{i(s-p)\psi_{jk}} e^{-is(\beta_j - \beta_k)} \\ &+ i \sum_{q=1}^{\infty} \text{Ms}_q^{(1)}(u_k; q_k) \text{se}_q(u_k; q_k) \frac{1}{2} \sum_{s,p=-\infty}^{\infty} (-1)^{(p-m+q-s)/2} B_s^q(q_k) A_p^m(q_j) \text{H}_{-p+s}(k_0 R_{jk}) e^{i(s-p)\psi_{jk}} e^{-is(\beta_j - \beta_k)} \end{aligned} \quad (5)$$

$$\begin{aligned} & i \text{Ms}_m^{(3)}(u_j; q_j) \text{se}_m(v_j; q_j) \\ &= \sum_{q=0}^{\infty} \text{Mc}_q^{(1)}(u_k; q_k) \text{ce}_q(u_k; q_k) \frac{1}{2} \sum_{s,p=-\infty}^{\infty} (-1)^{(p-m+q-s)/2} A_s^q(q_k) B_p^m(q_j) \text{H}_{-p+s}(k_0 R_{jk}) e^{i(s-p)\psi_{jk}} e^{-is(\beta_j - \beta_k)} \\ &+ i \sum_{q=1}^{\infty} \text{Ms}_q^{(1)}(u_k; q_k) \text{se}_q(u_k; q_k) \frac{1}{2} \sum_{s,p=-\infty}^{\infty} (-1)^{(p-m+q-s)/2} B_s^q(q_k) B_p^m(q_j) \text{H}_{-p+s}(k_0 R_{jk}) e^{i(s-p)\psi_{jk}} e^{-is(\beta_j - \beta_k)} \end{aligned} \quad (6)$$

where A and B in Eqs. (5) and (6) are the expansion coefficients for the even and odd periodic Mathieu functions respectively and H_m is the m th order Hankel function.

After introducing Eqs. (5) and (6) into Eq. (4), the total diffraction potential due to the wave scattering by all bodies can be expressed in terms of the local elliptical system of an arbitrarily selected body k . Thus, the boundary condition (1) can be directly applied. Apparently the potential in Eq. (1) should be obtained as the summation of the total diffraction potential and the incident wave potential given by Eq. (3). The application of the kinematical condition on the wetted surface of the body k will result in a complex truncated linear system which gives the unknown Fourier coefficients $\tilde{A}_m^{(k)}$ and $\tilde{B}_m^{(k)}$. It should be noted that the associated procedure is very lengthy and difficult and its details are omitted from the present short text. It should be also noted that it requires a major computational effort.

4 Numerical results

The method outlined in the present was applied for two parallel identical elliptical cylinders with a vertical imaginary connection arm between the centers (Fig. 2). With reference to the geometrical definitions of Fig. 1, the following dimensions were employed: $b_k/a_k=0.25$, $h/a_k=1.5$ and $R_{jk}/a_k=2$ for $k=1,2$. For validating the present analytical method, the results for the exciting forces were compared with numerical data obtained through a three – dimensional integral equation formulation [6] based on the distribution of pulsating sink – sources [7] on the wetted surface of the bodies. Indicative results are depicted in Figs. 3-6. Figs. 3 and 4 show the exciting forces in surge direction for 0 and 60 degrees of heading respectively, while Figs. 5 and 6 show the exciting forces in sway for 60 and 90 degrees of heading, respectively. All values are given in nondimensional form with normalization parameter $\rho g a_k^2 (H/2)$ where ρ is the water density.

It is immediately apparent that the comparisons between the present analytical approach and the panel method are very favorable which implies the accuracy of the semi-analytical method. In addition, the present method is extremely faster as it provides results within seconds. With regard to the variation of the exciting forces and the contribution of the hydrodynamic interactions, one can easily confirm that the recurrent scattering of waves has a profound effect for wave heading angles different than zero. The latter case is the only that can be approximated with sufficient accuracy by the single body case. For the 60 and 90 degrees of heading, the hydrodynamic interactions result in strong variations for the exciting forces, practically along the complete wave frequency range. The variations are stronger at higher wave frequencies where the magnitudes of the forces decay progressively. The exciting force in sway is the dominant part of the total hydrodynamic loading. In the relevant graphs (Figs. 5 and 6), it is interesting to observe that there is a specific range where the sway forces on both bodies exceed the forces on the single body. The associated region is located at the area where the maximum loading occurs. An additional important finding that can be drawn by inspecting the diagrams of the sway forces is that the lee side body (body 2) is not virtually protected as someone would expect. This is manifested in both non-zero heading cases examined in the present although the vertical distance between the elliptical cylinders is sufficiently large and equal to the length of the bodies. The hydrodynamic interactions are more influential for 90 degrees of heading, namely in the case where the second body (body 2) should be supposedly 'protected'.

5 References

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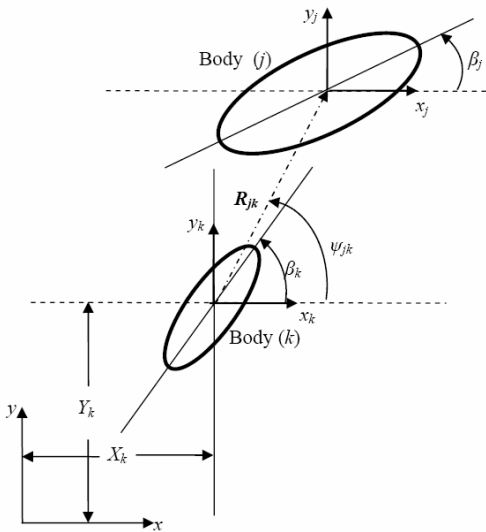


Fig. 1. General arrangement: coordinate systems and geometrical definitions.

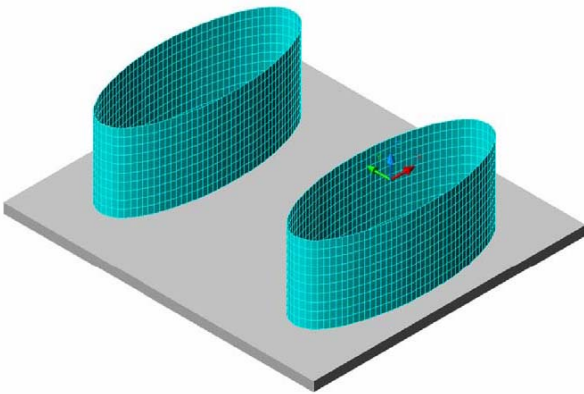


Fig. 2. Panel discretization of the two elliptical cylinders. Each mesh was built using 1000 elements.

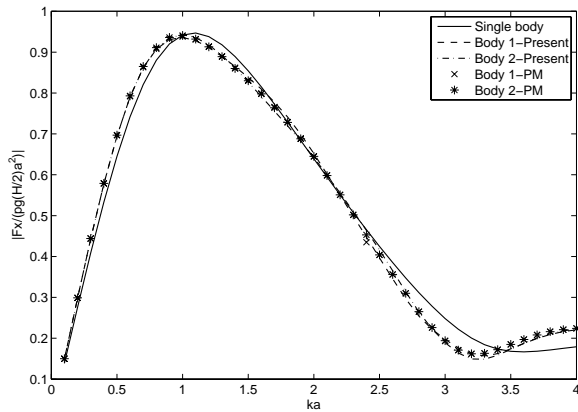


Fig. 3. Exciting forces in surge. Heading angle 0°.

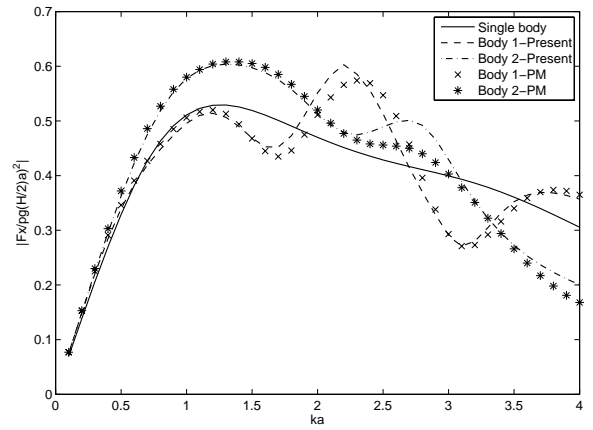


Fig. 4. Exciting forces in surge. Heading angle 60°.

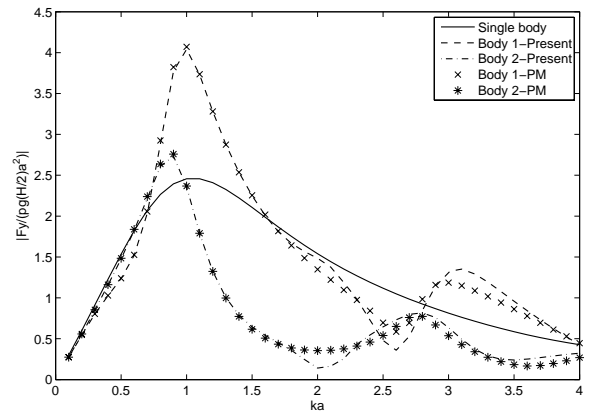


Fig. 5. Exciting forces in sway. Heading angle 60°.

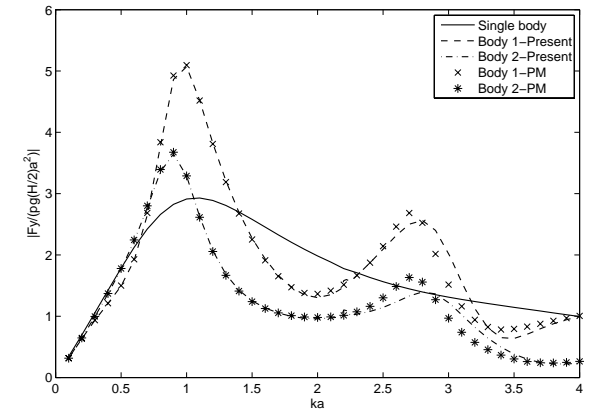


Fig. 6. Exciting forces in sway. Heading angle 90°.