NON-HARMONIC INTERNAL GRAVITY WAVE PACKETS IN STRATIFIED MEDIA

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Abstract. We consider the problem of reconstructing non-harmonic wave packets of the internal gravity waves generated by a source moving in a horizontally stratified medium. The solution is proposed in terms of modes, propagating independently at the adiabatic approximation, and described as a non-integral degree series of a small parameter characterizing the stratified medium.
Keywords: internal gravity waves; stratified fluid

INTRODUCTION

In this study we analyze the evolution of non-harmonic wave packets of internal gravity waves generated by a moving source under the assumption that the parameters of a vertically stratified medium (e.g. an ocean) vary slowly in the horizontal direction and in time, as compared to the characteristic length of the density $\rho(x, y, z)$. Specifically, we assume that a point source is moving with the supercritical velocity V along the x-axis at the depth z_0 in a layer $-H(\varepsilon x, \varepsilon y) < z < 0$ (ε is a small parameter) of the stratified fluid with the Brunt-Vaisala frequency $N(z)$. Using the Boussinesq approximation, the dynamics is defined by

$$\frac{\partial^2}{\partial t^2}\left(\Delta + \frac{\partial^2}{\partial z^2}\right)W + N^2(z)\Delta W = \delta''_{tt}(x + Vt)\delta(y)\delta'(z - z_0) \quad (1)$$

$$\Delta U + \nabla\frac{\partial W}{\partial z} = \delta(z - z_0)\nabla(\delta(x + Vt)\delta(y)) \quad (2)$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \quad (3)$$

where $U = (U_1, U_2)$ and W are the horizontal and vertical velocities, respectively. A specific form of the wave packets, which can be parameterized in terms of model functions, e.g. Airy functions or Fresnel functions, depends on a local behavior of the dispersion curves of individual modes in the vicinity of corresponding critical points.

METHODS

We modified the space-time ray method, which belongs to the class of geometrical optics methods. The key point of the proposed technique is the possibility to derive the asymptotic representation of the solution in terms of a non-integral degree series of the small parameter $\varepsilon = \lambda / L$, where λ is the characteristic wave length, and L is the characteristic scale of the horizontal heterogeneity. Specifically, we are looking for a solution as the sum of modes propagating independently (the adiabatic approximation), namely

$$W = A(\varepsilon x, \varepsilon y, z, \varepsilon t) R_0(\sigma) + \varepsilon^a B(\varepsilon x, \varepsilon y, z, \varepsilon t) R_1(\sigma) + \ldots \quad (4)$$

$$U = U_0(\varepsilon x, \varepsilon y, z, \varepsilon t) R_1(\sigma) + \ldots \quad (5)$$

$$R'_{i+1}(\sigma) = R_i(\sigma) \quad \sigma \equiv \left(\frac{S(\varepsilon x, \varepsilon y, \varepsilon t)}{a\varepsilon}\right)^a \quad (6)$$

where σ is on the order of one, and the functions $S(\varepsilon x, \varepsilon y, \varepsilon t)$, $A(\varepsilon x, \varepsilon y, z, \varepsilon t)$ and $U_0(\varepsilon x, \varepsilon y, z, \varepsilon t)$ are to be found [1]. Depending on the presence of a uniform (non-stratified) sublayer, the function $R_0(\sigma)$ is expressed in terms of Airy functions (Airy wave) or the Fresnel integrals (Fresnel wave).

The explicit form of the asymptotic solution was determined based on the principles of locality and asymptotic behavior of the solution in case of a stationary and horizontally homogeneous medium. First, $U_0(\varepsilon x, \varepsilon y, z, \varepsilon t)$ can be estimated with the $\varepsilon^{3/2}$-order of accuracy as

$$U_0 = -\frac{\partial A}{\partial z}\sqrt{2S}\frac{\nabla S}{\varepsilon\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right)} + O(\varepsilon^{3/2}) \tag{7}$$

The wave packet phase is calculated from the corresponding eikonal equations that are numerically solved along the characteristic curves. Specifically, the eikonal equation is defined by

$$\frac{\partial^2 A}{\partial z^2} + |\mathbf{k}|^2\left(\frac{N^2(z)}{\omega^2}\right)A = 0 \tag{8}$$

$$\mathbf{k}(\omega, x, y) = -\nabla S \quad \omega = \frac{\partial S}{\partial t} \tag{9}$$

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = |\mathbf{k}|^2 \tag{10}$$

The wave packet amplitudes are determined from the energy conservation laws along the characteristic curves. We represent $A(x, y, z, t)$ in the form $A(x, y, z, t) = \Psi(x, y, \omega(x, y, t)) f(x, y, z, \omega(x, y, t))$ where the function f is a normalized eigenfunction for the problem (8)-(9). Then, as can be shown, the function $\Psi(x, y, \omega(x, y, t))$ is determined from the energy conservation law

$$\frac{d}{dt}\ln\left(D\Psi^2\frac{\partial K}{\partial \omega}K^{-1}\right) = 0 \tag{11}$$

where $K(x, y, \omega) = |\mathbf{k}|^2$ and D is the Jacobian determinant to define transformation from the ray coordinates into the Cartesian ones.

A typical assumption made in studies on the internal wave evolution in stratified media is that the wave packets are locally harmonic. Our modification of the geometrical optics method, based on an expansion of the solution by model functions, allows us to describe the wave field structure both far from and at the vicinity of the wave front. This work solved the problem of describing the evolution of the non-harmonic packets of the internal gravity waves in a layer of an arbitrary stratified medium of varying depth with a non-stationary, horizontally non-uniform density.

Using the asymptotic representation of the wave field at a large distance from a source moving in a layer of constant depth, we solve the problem of constructing the uniform asymptotics of the internal waves in a medium of varying depth. The solution is obtained by modifying the previously proposed "vertical modes-horizontal rays" method, which avoids the assumption that the medium parameters vary slowly in a vertical direction. The solution is parameterized through the Airy or Fresnel waves. This allows us to describe not only the evolution of the non-harmonic wave packets propagating over a slow-varying fluid floor, but also specify the wave field structure associated with an individual mode both far from and close to the wave front of the mode. The Airy and Fresnel function argument is determined by solving the corresponding eikonal equations and finding vertical spectra of the internal waves. The wave field amplitude is determined using the energy conservation law, or another adiabatic invariant, characterizing wave propagation along the characteristic curves.

Modeling typical shapes and stratification of the ocean shelf, we obtain the analytic expressions describing the characteristic curves and examine characteristic properties of the wave field phase structure. We show that it is possible to observe some peculiarities in the wave field structure, depending on the shape of ocean floor, water stratification and the trajectory of a moving source. In particular, we analyze a spatial blocking effect of the low-frequency components of the wave field, generated by a source moving alongshore with a supercritical velocity. In addition, we construct the asymptotic representation of non-harmonic wave packets propagating in a medium with horizontally inhomogeneous and nonstationary density. Numerical analyses that are performed using typical ocean parameters reveal that actual dynamics of the internal gravity waves are strongly influenced by nonstationarity and horizontal inhomogeneity.

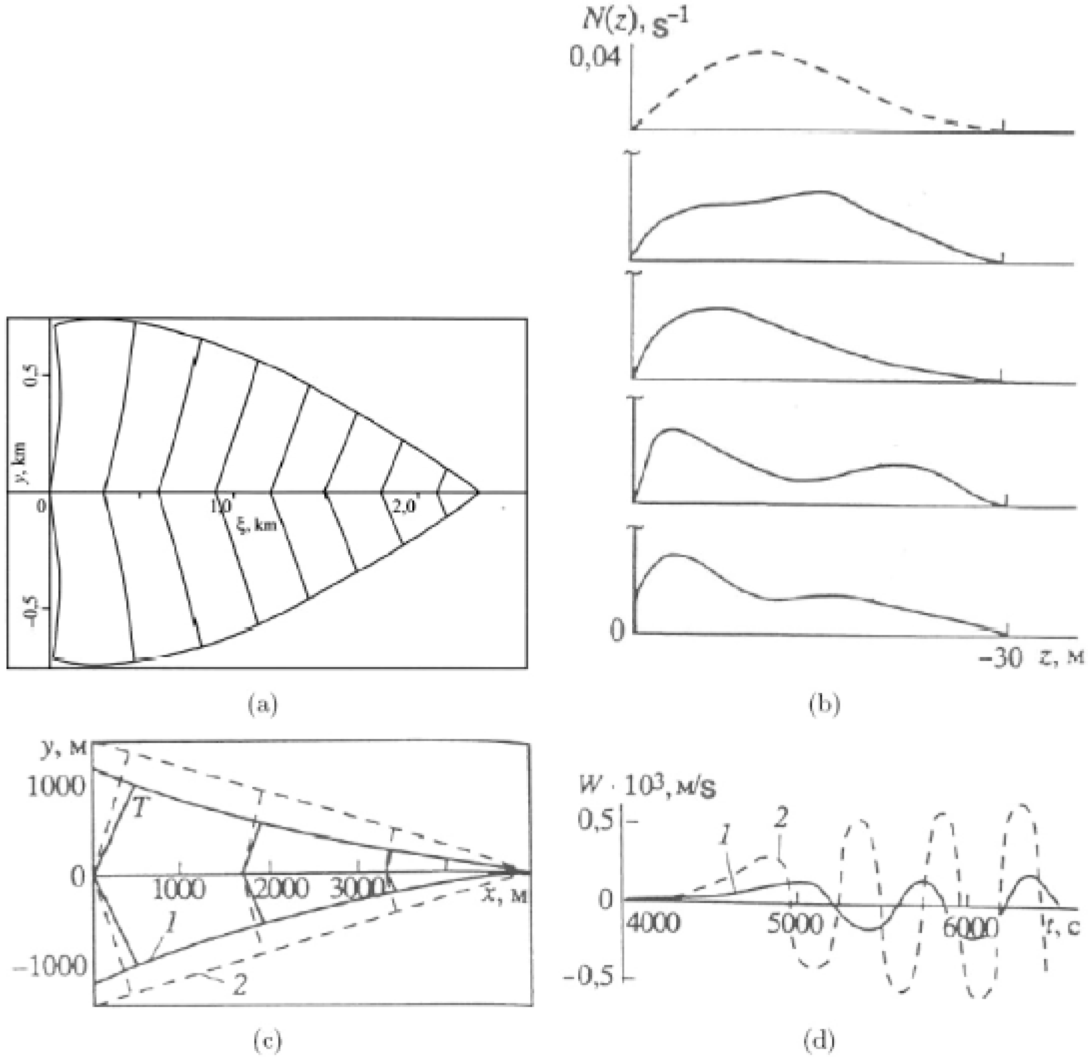

FIGURE 1. To illustrate the proposed techniques, two problems are considered: (a) the wave field at far distances in the vicinity of the separate mode front generated by a point source moving over a smoothly-varying floor; (b)-(d) the propagation of the Airy and Fresnel internal waves in a non-stationary stratified medium. Specifically, illustrated is the following: (a) constructed rays and wave fronts of non-harmonic wave packets for a typical ocean stratification and floor shape; (b) non-stationary stratification of the Black sea, shown with the time lag of 4 hours; (c,d) rays and wave fronts (c) and the amplitude (d) of the first mode in a non-stationary (solid line) and stationary (dashed line) case.

APPLICATIONS

To illustrate the proposed techniques, we consider two problems. The first one is to describe the wave field at far distances in the vicinity of the separate mode front generated by a point source moving over a smoothly-varying floor. The problem is solved by the "moving wave" method, which is a modification of the geometrical optics approach. The problem is considered at the weakly disperse approximation, when the solution describes the wave field only near to the corresponding wave front (local asymptotics). Fig. 1(a) gives an example illustrating the numerically reconstructed rays and wave fronts of non-harmonic wave packets for a typical ocean stratification and floor shape.

Second, we solve the propagation of the Airy and Fresnel internal waves in a non-stationary stratified medium, *e.i.* an ocean wherein the density $\rho(x, y, z, t)$ is a function not only of the coordinates but also of t. There are different scales of hydrophysical field variability in the ocean, ranging from small-scale variability with the time period up to 10 minutes, to meso-scale variability (24 hours), and synoptic and global variability with the time period from months up to a few years. We consider the non-stationary media with cycling parameters of about 24 hours and more. This allows us to use the geometrical optics approximation as the internal wave period is about 10 minutes or less. For the purpose of numerical calculations, the variability of the Brunt-Vaisala frequency $N(z, t)$ is assumed to be the same as that of the Black sea. Fig. 1(b) illustrates the Brunt-Vaisala frequency profile at different points of time with the time lag of 4 hours. Fig. 1(c) and 1(d) show the numerical reconstruction of the ray, wave fronts and the amplitude of the first mode in a non-stationary (solid line) and stationary (dashed line) case.

REFERENCES

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