An approximation to wave scattering by an ice polynya

L. G. Bennetts[1], T. D. Williams[2] and V. A. Squire[3]

[1]University of Otago, P.O. Box 56, Dunedin, New Zealand; e-mail: lbennetts@maths.otago.ac.nz

[2]University of Bristol, University Walk, Bristol, BS8 1TW, UK; e-mail: timothy.williams@bris.co.uk

[3]University of Otago, P.O. Box 56, Dunedin, New Zealand; e-mail: vernon.squire@otago.ac.nz

Introduction

Polynyas are openings or lakes in sea-ice that are of considerable current interest because of their probable proliferation due to global warming. This paper focuses on polynyas located in continuous ice, which are stable features that are common near the coast of Antarctica and off the Arctic land masses where they are created by offshore winds or upwelling or a combination of both. They also frequently occur in the near-continuous sea-ice of the Arctic Basic when divergent natural stresses open up the ice canopy.

Close to the open ocean the ice-cover is composed of relatively small floes and is highly energetic, as waves constantly fracture, break-up and redistribute it. Further into the region of ice-covered fluid the structure is more stable, as the wave energy there has reduced sufficiently to allow floes to exist at vast sizes and become near continuous. In this region and for greater penetrations, wave energy induces sea-ice to bend and flex rhythmically, with the subsequent oscillations at the fluid-ice interface being described as flexural-gravity waves. When a flexural-gravity wave interacts with a polynya it will simultaneously disturb the free-surface it encloses and be scattered away from the polynya into the surrounding ice-covered fluid region. Mathematical modelling of scattering by polynyas is particularly important, as they offer the best current possibility for performing in-situ field tests to validate theory. Such data are notably lacking from the majority of research in this area.

At present, theoretical work on the scattering by ice floes or irregularities in ice sheets is largely focussed on solving problems involving only two spatial-dimensions (see Squire, 2007). Due to the existence of analytical techniques such as Wiener-Hopf or residue calculus, a lot of progress can be made before resorting to numerical methods. Other more numerical approaches, such as those based on mode-matching, are also very efficient and are extremely easy to set up.

It is widely recognised that including the third spatial-dimension is necessary to properly simulate scattering in regions of sea-ice. However, this generally makes an analytical solution impossible to find and greatly increases the difficulty involved in obtaining a numerical solution. Notable work on scattering by three-dimensional geometries includes Meylan (2002) for finite floes and Porter and Evans (2007) for finite cracks.

In this paper we consider the scattering of flexural-gravity waves by a polynya of arbitrary shape. The solution method that we employ involves an approximation of the vertical motion by the single-mode that supports propagating waves. Inserting this into a variational principle eliminates the depth dependence and reduces the full three-dimensional boundary-value problem to a small system of two-dimensional Helmholtz equations. For more simple geometries this method has been shown to give high accuracy and it can be easily extended to the full-linear solution at the cost of increased numerics (see Bennetts et al., 2007). The two-dimensional system that remains for the polynya is reformulated via Green's theorem into a set of one-dimensional integro-differential equations that are solved using a Galerkin scheme.

Preliminaries and approximation

Consider a three-dimensional geometry that is comprised of a fluid domain of finite depth and that extends to infinity in all horizontal directions, and an ice-sheet of constant thickness D that covers it. Let the Cartesian coordinates $\mathbf{x} = (x, y)$ define the horizontal plane and let z be the vertical coordinate, which points upwards and has its origin set to coincide with the equilibrium surface of the fluid. The ice covering exists at all points $\mathbf{x}$ except for within a finite region $\mathbf{x} \in \Omega_0$ representing a polynya, in which the fluid surface is unloaded. We allow the shape of the polynya to be arbitrary, although we suppose that it is enclosed by the smooth contour $\Gamma = \Gamma(s)$ $(-L < s < L)$, with outward normal vector $\mathbf{n} = (\cos\Theta, \sin\Theta)$ and tangential vector $\mathbf{s} = (-\sin\Theta, \cos\Theta)$ $(\Theta = \Theta(s))$. If the bed depth is denoted by h and the equilibrium submergence of the ice by d then the fluid domain occupies $z \in (-h, 0)$ for $\mathbf{x} \in \boldsymbol{\Omega_0}$ and $z \in (-h, -d)$ for $\mathbf{x} \notin \boldsymbol{\Omega_0}$. Both h and d are assumed to be constant and we set $d = \rho_i/(\rho_w D)$, where $\rho_i = 922.5\,\mathrm{kg\,m^{-3}}$ and $\rho_w = 1025\,\mathrm{kg~m^{-3}}$ are the densities of the ice and water respectively, to ensure that the ice is neutrally bouyant.

When the system is in motion the ice experiences small-scale oscillations and, assuming harmonic time dependence, the position of the underside of the sheet is given by $z = -d + \Re\mathrm{e}(W\mathrm{e}^{-\mathrm{i}\omega t})$. Here ω is a prescribed angular frequency and $W = W(\mathbf{x})$ is the *displacement function*, which is currently unknown. We also make the regular assumptions of linear motions so that the velocity field of the fluid may be retrieved from $(\partial_{\mathbf{x}}, \partial_z) \cdot$

$\Re\mathrm{e}((g/\mathrm{i}\omega)\hat{\Phi}(\mathbf{x},z)\mathrm{e}^{-\mathrm{i}\omega t})$, where the unknown function $\hat{\Phi} = \hat{\Phi}(\mathbf{x},z)$ is the *velocity potential* and the constant $g = 9.81\,\mathrm{m\,s^{-2}}$ denotes acceleration due to gravity.

At this point we choose to partition the solution into the respective ice-covered and ice-free fluid regions and to restrict the vertical motion in each region to the single mode that supports propagating waves therein. To do this we approximate the velocity potential using

$$\hat{\Phi}(\mathbf{x},z) \approx \phi(\mathbf{x})\cosh\{k(z+h)\}\ (\mathbf{x}\in\Omega_0), \quad \hat{\Phi}(\mathbf{x},z) \approx \psi(\mathbf{x})\cosh\{\kappa(z+h)\}\ (\mathbf{x}\in\Omega_1), \tag{1}$$

where $\Omega_1 = \mathbb{R}^2\backslash\Omega_0$ defines the region of ice-covered fluid. This leaves us to find the function $\phi = \phi(\mathbf{x})$ in Ω_0 and $\psi = \psi(\mathbf{x})$ in Ω_1, both of which exist only in the horizontal plane. It is possible to extend the approximation (1) to gain the full-linear solution to an arbitrary degree of accuracy by adding in the modes that support evanescent waves and this work is currently in preparation.

In the above, the quantity k is the propagating wavenumber within the polynya, and is calculated as the positive, real root of the free-surface dispersion relation $k\tanh(kh) = \sigma$, where $\sigma = \omega^2/g$. Similarly, κ is the propagating wavenumber when the fluid is ice-covered, which is the only positive real root of the ice-covered dispersion relation

$$(1 - \sigma d + \beta\kappa^4)\kappa\tanh(\kappa H) = \sigma, \tag{2}$$

where $H = h - d$ is the fluid depth beneath the ice. In equation (2) the properties of the ice appear through d and $\beta = ED^3/12(1-\nu^2)\rho_w g$, which is a scaled version of the flexural rigidity of the ice; in addition, $E = 5\times10^9$ Pa and $\nu = 0.3$ are the Young's modulus and the Poisson's ratio respectively.

The system of ice and fluid is forced by a plane incident wave that propagates towards the polynya from the far-field $|\mathbf{x}| \to \infty$ at the oblique angle $\vartheta \in (0, 2\pi)$ with respect to the x-axis. Due to our choice to base our approximation on the vertical mode that supports propagating waves, the incident wave can be represented exactly in Ω_1 and we write it as $\psi_I\cosh\{\kappa(z+h)\}$ where $\psi_I = \mathrm{e}^{\mathrm{i}\kappa(x\cos\vartheta + y\sin\vartheta)}$.

Combining the approximation (1) with a variational principle, which is equivalent to the governing equations of the full-linear problem, produces a set of equations to be satisfied by the functions ϕ and ψ. Within the polyna this dictates that, if we denote $\nabla \equiv (\partial_x, \partial_y)$, the function ϕ satisfies the the second-order differential equation

$$\nabla^2\phi + k^2\phi = 0 \quad (\mathbf{x}\in\Omega_0). \tag{3a}$$

In the region of ice-covered fluid the variational principle also creates an approximation of the displacement function $w \approx W$ indirectly through its association to the velocity potential. We write the equations to be satisfied by ψ and w as the second-order system

$$\nabla^2\Psi + CK^2C^{-1}\Psi = \mathbf{0} \quad (\mathbf{x}\in\Omega_1), \tag{3b}$$

where the vector of unknowns is $\Psi = (\psi, w, \hat{w})^T$ and we have introduced the notation $\hat{w} \equiv \beta\nabla^2 w$ for convenience. In equation (3b) the matrix $K = \mathrm{diag}\{\kappa, \mu_1, \mu_2\}$ contains the propagating wavenumber in the ice, κ, and values μ_j that define (typically) oscillatory-evanescent waves scattered by the polynya. These quantities are calculated as the roots μ of the quartic equation $(\beta\mu^4 + 1 - \sigma d) + 2\beta\kappa\sinh(2\kappa H)(\kappa^2+\mu^2)/a_+ = 0$ that exist in the upper-half plane, where the constant $a_+ = \int_{-h}^{-d}\cosh^2\{\kappa(z+h)\}\,\mathrm{d}z$. The matrix C contains the eigenvalues of the system and is defined as $C = [\mathbf{c}(\kappa),\ \mathbf{c}(\mu_1),\ \mathbf{c}(\mu_2)]$ where $\mathbf{c}(u) = (1, u\sinh(uH)/\sigma, -\beta u^3\sinh(uH)/\sigma)^T$.

The approximate velocity potentials are linked at the boundary Γ by the jump conditions

$$p_-\phi = p_+\psi, \quad (a_-/p_-)\partial_n\phi = \partial_n\psi, \tag{4}$$

where the known values $p_- = \int_{-h}^{-d}\cosh\{k(z+h)\}\cosh\{\kappa(z+h)\}\,\mathrm{d}z$ and $a_- = \int_{-h}^{0}\cosh^2\{k(z+h)\}\,\mathrm{d}z$, and $\partial_n \equiv \mathbf{n}\cdot\nabla$ is the normal derivative. The ice must also experience no bending moment or shearing stress at the edge of the polynya, and this is expressed as

$$\hat{w} - (1-\nu)\beta(\partial_s^2 + \Theta'\partial_n)w = 0, \quad \partial_n\hat{w} + (1-\nu)\beta\partial_s(\partial_s\partial_n - \Theta'\partial_s)w = 0, \tag{5}$$

respectively, where $\partial_s \equiv \mathbf{s}\cdot\nabla$ is the tangential derivative. In the far-field $|\mathbf{x}| \to \infty$ the regular Sommerfeld radiations condition must be applied to ψ.

Integral equations

In order to solve the problem posed by the differential equations (3a-b) we use Green's theorem to reformulate them into integral representations that may then be matched on their common boundary Γ. Beginning with the free-surface equations within the polynya, we may write

$$\epsilon\phi(\mathbf{x}) = \frac{1}{4\mathrm{i}}\int_\Gamma \{(\partial_N \mathrm{H}_0(k\mathbf{r}))\phi(\mathbf{X}) - \mathrm{H}_0(k\mathbf{r})(\partial_N\phi(\mathbf{X}))\}\,\mathrm{d}S \quad (\mathbf{x}\in\Omega_0), \tag{6a}$$

in which H_0 denotes the Hankel function of the first kind of order 0 and the quantity ϵ is defined as $\epsilon = \frac{1}{2}$ $(\mathbf{x}\in\Gamma)$ and $\epsilon = 1$ $(\mathbf{x}\notin\Gamma)$. Our integration parameters are defined through the source variable S, which corresponds to s, so that the curve $\Gamma = \Gamma(S)$, and the coordinate vector $\mathbf{X} = \mathbf{X}(S) \in \Gamma$ then corresponds to the Cartesian coordinates $\mathbf{x}$. The normal derivative of the source variables is ∂_N and $\mathbf{r} \equiv |\mathbf{x} - \mathbf{X}|$ defines the distance between the field and source variables.

Similarly, outside of the polynya, in which the velocity potential is coupled to the interfacial displacement we derive the representation

$$\epsilon\Psi(\mathbf{x}) = \Psi_I(\mathbf{x}) - \frac{1}{4\mathrm{i}}C\int_\Gamma \{(\partial_N\mathcal{H}_0(K\mathbf{r}))C^{-1}\Psi(\mathbf{X}) - \mathcal{H}_0(K\mathbf{r})C^{-1}(\partial_N\Psi(\mathbf{X}))\}\,\mathrm{d}S \quad (\mathbf{x}\in\Omega_1), \tag{6b}$$

where the diagonal matrix $\mathcal{H}(K\mathbf{r}) = \mathrm{diag}\{\mathrm{H}_0(\kappa\mathbf{r}), \mathrm{H}_0(\mu_1\mathbf{r}), \mathrm{H}_0(\mu_2\mathbf{r})\}$ and the vector $\Psi_I = C(\psi_I, 0, 0)^T$ contains the contribution of the incident wave.

Now, letting $\mathbf{x}\to\Gamma$ in the above integral representations and applying the jump conditions (4) gives rise to a system of integral equations, which we write as

$$u(s) = \int_\Gamma \{m_{0,0}(s|S)u(S) - m_{0,1}(s|S)v(S)\}\,\mathrm{d}S, \quad \mathbf{u}(s) = \mathbf{u}_I(s) - \int_\Gamma \{M_{1,0}(s|S)\mathbf{u}(S) - M_{1,1}(s|S)\mathbf{v}(S)\}\,\mathrm{d}S. \tag{7}$$

These equations are to be solved for the vectors of functions $\mathbf{u} = (u, [w]_\Gamma, [\hat{w}]_\Gamma)^T$ and $\mathbf{v} = (v, [\partial_n w]_\Gamma, [\partial_n\hat{w}]_\Gamma)^T$ in which $u = p_-[\phi]_\Gamma = p_+[\psi]_\Gamma$ and $v = (a_-/p_-)[\partial_n\phi]_\Gamma = [\partial_n\psi]_\Gamma$, and where $[\cdot]_\Gamma$ denotes that the included quantity is evaluated on the contour Γ. We also define the scalar Kernels $m_{0,0}$ and $m_{0,1}$ as $2\partial_N\mathrm{H}_0(k|s-S|)$ and $2(p_-^2/a_-)\mathrm{H}_0(k|s-S|)$ respectively, and the 3×3 matrix Kernels $M_{1,0}$ and $M_{1,1}$ as $2\partial_N\mathcal{H}_0(K|s-S|)$ and $2P\mathcal{H}_0(K|s-S|)$ respectively, where $P = \mathrm{diag}\{p_+, 1, 1\}$. The system is forced by the vector $\mathbf{u}_I = 2P[\Psi_I]_\Gamma$.

Application of the bending moment and shearing stress conditions (5) allows the removal of $\hat{w}$ and $\partial_n\hat{w}$ from (7) and in-doing-so reduces the unknowns present in the above integral system to the appropriate number to match the equations. An integro-differential system remains to be solved for the functions u, v, w and $\partial_n w$, from which we may then obtain the corresponding values of $\hat{w}$ and $\partial_n\hat{w}$. Once the values of the unknown functions have been obtained on the boundary Γ, their values throughout the domain in which they exist may be found using the appropriate integral representation of (6).

In order to solve the system of equations that we have generated on the boundary Γ we invoke a Galerkin scheme with exponential basis functions. The unknowns are therefore expanded in terms of the orthogonal set $\{\chi_m : m\in\mathbb{R}\}$ where $\chi_m = \chi_m(s) = (1/2L)\mathrm{e}^{\mathrm{i}\lambda_m s}$ and $\lambda_m = m\pi/L$, so that $u(s) = \sum u_m\chi_m(s)$, $v(s) = \sum u_v\chi_m(s)$, $w(s) = \sum w_m\chi_m(s)$ and $\partial_n w(s) = \sum\omega_m\chi_m(s)$. We then truncate this representation to the dimension $2M+1$ that achieves sufficient accuracy and solve for the constants u_m, v_m, w_m and ω_m $(m = -M,\ldots,M)$.

By taking the inner-product of the equations with each of the basis functions χ_m $(m = -M,\ldots,M)$ in turn, we derive the set of coupled equations

$$u_n = \sum_{m=-M}^{M}\{\langle\langle m_{0,0}\rangle\rangle_{m,n}u_m - \langle\langle m_{0,1}\rangle\rangle_{m,n}v_m\}, \;\; \tilde{\mathbf{u}}_n = \mathbf{u}_{I,n} - \sum_{m=-M}^{M}\{\langle\langle M_{1,0}\rangle\rangle_{m,n}\tilde{\mathbf{u}}_m - \langle\langle M_{1,1}\rangle\rangle_{m,n}\tilde{\mathbf{v}}_m\} \tag{8}$$

for $n = -M,\ldots,M$, where $\langle\langle F\rangle\rangle_{m,n} = \int_{-L}^{L}\int_{-L}^{L} F(s|S)\chi_m(S)\overline{\chi}_n(s)\,\mathrm{d}S\,\mathrm{d}s$. Within these equations we have the bending moment and shearing stress conditions embedded through our definitions of $\tilde{\mathbf{u}}_n$ and $\tilde{\mathbf{v}}_n$ which are $(u_n, w_n, -(1-\nu)\beta\{\lambda_n^2 w_n - \sum_m\langle\Theta'\rangle_{m,n}\omega_m)^T$ and $(v_n, \omega_n, -(1-\nu)\beta\{\lambda_n^2\omega_n - \sum_m\langle\Theta'' - \mathrm{i}\lambda_n\Theta'\rangle_{m,n}w_m)^T$ respectively, where $\langle f\rangle_{m,n} = \int_{-L}^{L} f(S)\chi_m(S)\overline{\chi}_n(S)\,\mathrm{d}S$.

Equation (8) represents a linear system of $4(2M+1)$ equations to be solved for the constants u_m, v_m, w_m and ω_m that define the Galerkin approximations. For a smooth curve, such as Γ, all of the inner-products needed to form these equations incorporate only integrable functions and they may therefore be straightforwardly calculated by numerical means. It is also possible to consider non-smooth curves by using integration by-parts to eliminate the singularities that are introduced. This extension will appear in a forthcoming work.

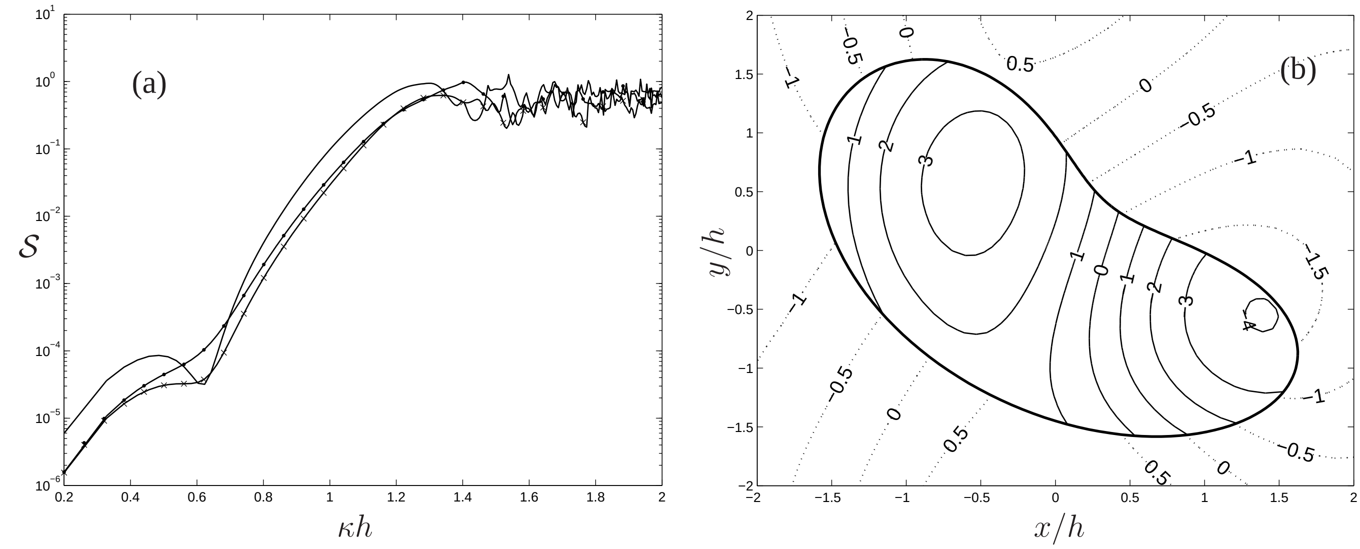

Figure 1: Part (a) shows the scattering cross-section $\mathcal{S}$ against non-dimensional wavenumber κh for three ellipses. Part (b) is a contour plot of the response of a polynya to an incident wave of non-dimensional length $100/h$. The elevation of the free-surface within a polynya (solid contours) and the diffracted wave in the surrounding ice-covered fluid (broken contours) are both plotted.

Numerical results

The above figures display two sets of numerical results produced using the theory outlined in this work. All spatial parameters are non-dimensionalised with respect to the depth h and in both sets of results we fix $D/h = 0.05$.

For the first set of results we consider the far-field response induced by elliptical polynyas. Using the asymptotic behaviour of Hankel functions for large arguments in expression (6b), we may deduce that, in terms of the polar coordinates (r, θ), the diffracted wave may be written as $\psi - \psi_I \sim (2/\pi\kappa r)^{1/2}\mathrm{e}^{\mathrm{i}\kappa(r-\pi/4)}\mathcal{F}(\theta)$ as $r \to \infty$. Here, the function $\mathcal{F}$ is known as a *diffracted far-field amplitude* and is readily calculated.

Figure 1(a) displays the *scattering cross-section* $\mathcal{S} = (1/2\pi)\int_{-\pi}^{\pi} |\mathcal{F}|^2 \,\mathrm{d}\theta$ as a continuous function of the non-dimensional wavenumber κh for three elliptical polynyas, where the incident wave propagates at an angle $\vartheta = 30°$. The ellipses are defined by $\mathbf{x}/0.2h = (a\cos(s\pi/L),\, b\sin(s\pi/L))$, where the ratio a/b is chosen to define the eccentricity. The unmarked curve denotes the results for a circle $(a = b = 1)$, the dotted curve the ellipse with $(a, b) = (1, 0.25)$ and the curve with crosses $(a, b) = (0.25, 1)$.

For low frequencies, although the three curves are similar there is a striking difference in their structure around the point $\kappa h \approx 0.6$, at which the minimum appearing for the circular polynya compares to the inflections seen for the elliptical polynyas. Away from this feature, as would be expected, the circular polynya generally produces a diffracted wave of a larger amplitude. The values of the scattering cross-section grow with wavenumber until $\kappa h \approx 1.25$, at which point they level out and experience the fine structure familiar in far-field responses at high frequencies.

Figure 1(b) gives a contour plot showing the reponse of a more unusual shape and one that better represents a real polynya. This is defined by the curve $\mathbf{x}/0.2h = (r\cos(s\pi/2L), r\sin(s\pi/2L))$, where the radial coordinate is $r = \cos^3(s\pi/2L) + \sin^3(s\pi/2L)$, and it is shown by the thick line on the figure.

In the case depicted, the incident wave is of non-dimensional length $100/h$ and is set to unit amplitude. Within the polynya, the solid contours denote the level surfaces of the wave elevation at $t = 0$, with the broken contours outside of the polynya, likewise the elevation of the surrounding fluid-ice interface induced by the diffracted wave at $t = 0$. Only 21 basis functions $(M = 10)$ were required to ensure convergence and energy conservation.

Acknowledgements

Supported by the Marsden Fund Council from Government funding administered by the Royal Society of New Zealand and by the Otago University. TDW holds a New Zealand Science and Technology Postdoctoral Fellowship.

References

Bennetts, L. G, Biggs, N. R. T, and Porter, D (2007). "A multi-mode approximation to wave scattering by ice sheets of varying thickness.," *J. Fluid Mech.*, Vol 579, pp 413–443.

Meylan, M. H (2002). "Wave response of an ice floe of arbitrary geometry," *J. Geophys. Res.*, Vol 107(C1), doi:10.1029/2000JC000713.

Porter, R and Evans, D. V (2007). "Diffraction of flexural waves by finite straight cracks in an elastic sheet over water," *J. Fluids. Struct.*, Vol 23(2), pp 309–327.

Squire, V. A (2007). "Of ocean waves and sea-ice revisited," *Cold Reg. Sci. Technol.*, Vol 49, pp 110–133.