Linear and Nonlinear Complementary Mild Slope Equations

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For a time harmonic refraction diffraction problem, Kim & Bai (2004) introduced a vector stream function:

$$\Psi[x, y, z] \equiv \int_{-h[x,y]}^{z} u[x, y, z_0]\, dz_0$$

where $\eta[x,y]$, u[x,y,z] and w[x,y,z] are the wave elevation, horizontal velocity and vertical velocity complex amplitudes, respectively.

The velocity field and wave elevation are given by:

$$u = \frac{\partial \Psi}{\partial z} \quad , \quad w = -\nabla \cdot \Psi,$$

$$\eta = \frac{1}{i\omega} \nabla \cdot \Psi[x, y, 0]$$

They showed that $\Psi[x,y,z]$ satisfies Laplace's Equation. Ψ clearly vanishes at z = -h. In two dimensions ($\Psi=(\psi,0)$) the formulation reduces to the well known scalar stream function formulation.

The solution was approximated by representing Ψ as:

$$\Psi[x, y, z] = \Psi_0[x, y]\, f[h, z] \quad , \qquad f[h, z] = \frac{\sinh[k[z+h]]}{\sinh[kh]}$$

where $\Psi_0[x, y]$ is the unknown to be solved. Through an Averaged Lagrangian they obtained a mild slope type equation, named the Complementary Mild Slope Equation (CMSE). & Bai (2004) studied this equation for one horizontal dimension (2D) and showed that it performed very well compared to similar models derived from the velocity potential formulation, such as the Modified MSE (Chamberlain and Porter, 1995) and the Extended MSE (Kirby 1986) which improved the original MSE (Berkhoff, 1972). The dominant role of resonant reflection of water waves in refraction and diffraction has become quite clear (Mei, 1985; Agnon and Pelinovsky, 2001)

Agnon (1999) has used Operational Calculus to derive a pseudo-differential equation (the Augmented Mild Slope Equation, AMSE) in terms of the velocity potential. This equation was used to study the theoretical accuracy of the Mild Slope Equation and its extensions, by deriving them as approximations to the AMSE. Here we derive an Augmented CMSE, and show that the CMSE is a high order approximation to the Augmented CMSE, which explains its excellent performance. We also extend the application of the CMSE to three dimensional problems and obtain a Nonlinear CMSE.

Following Agnon (1999) and Rayleigh (1876) we expand the stream function Ψ in a series about Ψ_0, using the Laplace equation (and the combined free surface boundary condition) to replace even ordered vertical derivatives by corresponding powers of the horizontal Laplace Operator, we write:

$$\Psi[z] = \mathrm{Exp}\left[z \frac{d}{dz}\right] \Psi_0 = F[z, \nabla]\, \Psi_0$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

and F is a differential operator of infinite order

$$F[z, \nabla] \equiv \left(\mathrm{Cos}[z\nabla] - \frac{1}{\sigma}\, \mathrm{Sin}[z\nabla]\, \nabla\right) \qquad \sigma \equiv k\, \mathrm{Tanh}[h\, k]$$

According to its definition, $0\Psi = 0$ at z = -h, that is

$F[-h, \nabla]\, \Psi_0 = 0$.

By adding $F[-h_0, \nabla]\, \Psi_0 - F[-h, \nabla]\, \Psi_0$ on both sides of the equation we get:

$$F[-h_0, \nabla]\, \Psi_0 = (F[-h_0, \nabla] - F[-h, \nabla])\, \Psi_0 \qquad (*)$$

We now define:

$$G[h_0, \nabla] \equiv \frac{\mu}{F[-h_0, \nabla]} = \frac{\mu}{\mathrm{Cos}[h_0\nabla] + \frac{1}{\sigma}\, \mathrm{Sin}[h_0\nabla]\, \nabla}$$

$\mu \equiv \nabla^2 + k_0^2$ is a small parameter standing for the detuning from resonance.

Operation of G on both sides of 0 (*) yields:

$$\mu\,\Psi_0 = G\,(F_0 - F)\,\Psi_0 \tag{**}$$

(**) is the Augmented CMSE.   We expand F, G in μ:

$$F[h, \nabla] = F[h, k_0] - F'[h, k_0]\,\mu\,,$$

$F_0[h_0, \nabla] = F_0 - F'_0\,\mu$ where $F_0 = F(h=h_0)$ and

$$G[h_0, \kappa] = \frac{\mu}{F[h_0, \nabla]} = -\frac{1}{F'_0} + \frac{1}{2}\,\frac{F''_0}{(F'_0)^2}\,\mu$$

Expansion of 0 to $O(\mu)$ gives:

$$\mu\,\Psi_0 = G\,(\,(F_0 - F) - (F'_0 - F')\,\mu)\,\Psi_0$$

which leads to:

$$\mu\,\Psi_0 = -\frac{1}{F'_0}\,(F_0 - F)\,\Psi_0 + \frac{1}{2}\,\frac{F''_0}{(F'_0)^2}\,(\nabla^2 + k_0^2)\,(F_0 - F)\,\Psi_0 + \frac{1}{F'_0}\,(F'_0\,\mu - F'\,\mu)\,\Psi_0 + O(\mu^2)$$

We now note that:

$$(F_0 - F)\,\Psi_0 = (F_0 - F)\,\nabla^2\Psi_0 = (F'_0 - F')\,\mu\,\Psi_0 = 0$$

$$\nabla F_0\,\Psi_0 = \nabla^2 F_0\,\Psi_0 = 0$$

so we get:

$$\nabla^2 F_0 - \nabla^2 F = -\frac{\partial F[h, k]}{\partial h}\,\nabla^2 h - \frac{\partial^2 F[h, k]}{\partial h^2}\,(\nabla h)^2$$

Finally we obtain:

$$\mu\,\Psi_0 = a\,\nabla h\,\nabla\Psi_0 + b\,\nabla^2 h\,\Psi_0 + c\,(\nabla h)^2\,\Psi_0 \tag{***}$$

$$a = -\frac{F''_0}{(F'_0)^2}\,\frac{\partial F}{\partial h},\quad b = -\frac{1}{2}\,\frac{F''_0}{(F'_0)^2}\,\frac{\partial F}{\partial h},\quad c = -\frac{1}{2}\,\frac{F''_0}{(F'_0)^2}\,\frac{\partial^2 F[h, k]}{\partial h^2}$$

This has the same form as the CMSE. Comparison of the coefficients of (***) with the coefficients of the CMSE shows that the first two coefficients a, b are identical to the corresponding coefficients. These two coefficients are associated with Class I Bragg resonance, as defined by Liu & Yue (1998). Thus, we see that the CMSE is accurate to order $\mu\delta$, where δ = h-h_0, while the Modified MSE has an error of order $\mu\delta$. This explains the excellent performance of the CMSE.

c, the last coefficient in (***) and the corresponding coefficient in the CMSE are related to the square of the bottom slope. They are associated with Class II Bragg resonance, and as such neither of them is accurate.

The application of the CMSE to three dimensional problems and the Nonlinear CMSE will also be discussed.

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