# On the stability of bifurcating solutions in some problems about capillary-gravity waves.

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In prolongation of our previous investigations on capillary-gravity surface waves in spatial layers (see [1-3] and bibliography to them) the problem about potential flows of incompressible heavy capillary floating fluid in a layer of infinite depth with free upper boundary is considered. Periodical with periods $\frac{2\pi}{a} = a_1$ and $\frac{2\pi}{b} = b_1$ potential flows of a heavy capillary floating deep fluid in spatial layer with free upper boundary close to horizontal plane $z = 0$ are bifurcating from the basis flow with constant velocity $V$ in $Ox$-direction. Velocity potential has form $\varphi(x, y, z) = Vx + \Phi(x, y, z)$. In dimensionless variables this problem is described by the system of differential equations

$$\Delta\Phi = 0, -\infty < z < f(x, y); \tag{1}$$

$$\frac{\partial\Phi}{\partial z} - \frac{\partial f}{\partial x} = (\nabla f, \nabla_{xy}\Phi) = \frac{\partial\Phi}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial f}{\partial y}, z = f(x, y); \tag{2}$$

$$\frac{\partial\Phi}{\partial x} + \frac{1}{2}|\nabla\Phi|^2 + F^2 f + \frac{k}{\sqrt{1 + |\nabla f|^2}}\left[F^2 + (-\nabla f \cdot \nabla_{xy} + \frac{\partial}{\partial z})(\frac{\partial\Phi}{\partial x} + \frac{1}{2}|\nabla\Phi^2|)\right] -$$

$$-\gamma F^2 div(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}) = const, z = f(x, y) \tag{3}$$

with decreasing conditions of the function $\Phi$ and its first derivatives on infinity. The second equality in (1)–(3) is a kinematic interfacial condition, and the third one describes the force balance (the Bernoulli integral), $F^2 = \frac{gL}{V^2}$ (the Froud number), $\gamma = \frac{\sigma}{\rho g L^2}$(the Bond number), $k = \frac{\rho_0}{\rho L}$.

The system (1)–(3) is invariant to the 2-dimensional shifts group $L_\beta g(x, y) = g(x + \beta_1, y + \beta_2)$ and the reflections

$$S_1\colon x \to -x, \Phi(x, y, z) \to -\Phi(-x, y, z), f(x, y) \to f(-x, y),$$
$$S_2\colon y \to -y, \Phi(x, y, z) \to \Phi(x, -y, z), f(x, y) \to f(x, -y)$$

The linearized system can be obtained by straightening free upper boundary — change of variables $\zeta = z - f(x, y)$, $\Phi(x, y, \zeta + f(x, y)) = u(x, y, \zeta)$ and setting $F^2 = F_0^2 + \varepsilon$

$$\Delta u = w^{(0)}(u, f), -\infty < \zeta < 0; \tag{4}$$

$$u_\zeta - f_x = w^{(1)}(u,f), \zeta = 0; \tag{5}$$

$$u_x + ku_{x\zeta} + F_0^2 - \gamma F_0^2 \Delta f = w^{(2)}(u,f,\varepsilon), \zeta = 0; \tag{6}$$

$$k < \gamma F_0^2, \quad \text{(the ellipticity condition of the Bernoulli integral (6))} \tag{7}$$

where $w^{(j)}$, $j = 0, 1, 2$ are small nonlinearities. The system (4)–(6) can be written as the nonlinear functional equation $BX = R(X,\varepsilon)$, $R(0,\varepsilon) \equiv 0$, $R_x(0,0) = 0$, where $X = \{u, f\}$ is the bifurcation point problem with Fredholm [4] operator $B = B_{mn}\colon C^{2+\alpha}(\Pi_0 \times (-\infty,0]) + C^{2+\alpha}(\Pi_0) \to C^\alpha(\Pi_0 \times (-\infty,0]) + C^\alpha(\Pi_0) + C^\alpha(\Pi_0)$, $0 < \alpha < 1$, $\Pi_0$ is the periodicity rectangle in $(x,y)$ plane.

Presenting the function $f(x,y)$ by its Fourier series

$$\sum_{m,n}(a_{mn}\cos max\cos nby + b_{mn}\cos max\sin nby +$$

$$+c_{mn}\sin max\cos nby + d_{mn}\sin max\sin nby),$$

in the homogeneous equation $BX = 0$ we find

$$u(x,y,\zeta) = \sum_{m,n}\frac{mae^{s_{mn}\zeta}}{s_{mn}}\left(c_{mn}\cos max\cos nby + d_{mn}\cos max\sin nby - \right.$$
$$\left. -a_{mn}\sin max\cos nby - b_{mn}\sin max\sin nby\right), s_{mn}^2 = m^2a^2 + n^2b^2, F_{mn}^2 = F_0^2.$$

Then the equation (6) gives the dispersion relation

$$\left(k + \frac{1}{s_{mn}}\right)m^2a^2 = F_0^2(1+\gamma s_{mn}^2), \quad s_{mn}^2 = m^2a^2 + n^2b^2, \quad F_{mn}^2 = F_0^2 \tag{8}$$

($m, n$ are positive integers, $n$ may be equal zero), at the satisfaction of which for some pairs $(m_j, n_j)$, $j = 1, 2, \ldots, \kappa$ the zero-subspace $N(B)$ of the linearized operator $B$ has form

$$\begin{aligned}
\hat\varphi_{1j} &= \{-v_{1j}(\zeta)\sin m_jax\cos n_jby, v_{2j}\cos m_jax\cos n_jby\},\\
\hat\varphi_{2j} &= \{-v_{1j}(\zeta)\sin m_jax\sin n_jby, v_{2j}\cos m_jax\sin n_jby\},\\
\hat\varphi_{3j} &= \{v_{1j}(\zeta)\cos m_jax\cos n_jby, v_{2j}\sin m_jax\cos n_jby\},\\
\hat\varphi_{4j} &= \{v_{1j}(\zeta)\cos m_jax\sin n_jby, v_{2j}\sin m_jax\sin n_jby\},
\end{aligned}$$

where $v_{1j}(\zeta) = \dfrac{m_ja\sqrt{ab}}{\pi s_{m_jn_j}}e^{s_{m_jn_j}\zeta}, v_{2j} = \dfrac{\sqrt{ab}}{\pi}.$

Solving the arising branching equations (BEqs) at the different $n = \dim N(B)$, we find its solutions (expressed by the coefficients of corresponding BEqs). We consider now the 4-dimensional branching. The orbital stability of the branching solutions (1)-(3) is determined [5] by the stability of stationary solutions of the equation $\frac{d\eta}{dt} = t(\eta,\varepsilon)$, where $t(\eta,\varepsilon)$ is the left part of branching system, $\varepsilon = F^2 - F_{mn}^2$. The stability of these ones is determined by the signs of eigenvalues of Jacobian matrix $J = \left[\frac{\partial \bar{t}_i}{\partial \eta_j}\right]$ on these solutions. The action of operator $L_{\beta_1\beta_2}$ on the arbitrary element $N(B_{mn})$ is equivalent to the transformation of its coordinates with the aid of matrix $A_g$ (here $f_1(\beta_1, \beta_2) = \cos ma\beta_1 \cos nb\beta_2, f_2(\beta_1, \beta_2) = \cos ma\beta_1 \sin nb\beta_2, f_3(\beta_1, \beta_2) = \sin ma\beta_1 \cos nb\beta_2, f_4(\beta_1, \beta_2) = \sin ma\beta_1 \sin nb\beta_2$)

$$A_g = \frac{\pi}{\sqrt{ab}} \begin{pmatrix} f_1(\beta_1, \beta_2) & f_2(\beta_1, \beta_2) & f_3(\beta_1, \beta_2) & f_4(\beta_1, \beta_2) \\ -f_2(\beta_1, \beta_2) & f_1(\beta_1, \beta_2) & -f_4(\beta_1, \beta_2) & f_3(\beta_1, \beta_2) \\ -f_3(\beta_1, \beta_2) & -f_4(\beta_1, \beta_2) & f_1(\beta_1, \beta_2) & f_2(\beta_1, \beta_2) \\ f_4(\beta_1, \beta_2) & -f_3(\beta_1, \beta_2) & -f_2(\beta_1, \beta_2) & f_1(\beta_1, \beta_2) \end{pmatrix}.$$

By using the matrix $A_g$ the family of solutions is determined $\tilde{\eta} = A_g \tilde{\eta}_0(\varepsilon) = \frac{\pi}{\sqrt{ab}}(f_1(\beta_1, \beta_2), -f_2(\beta_1, \beta_2), -f_3(\beta_1, \beta_2), f_4(\beta_1, \beta_2))^T \left(-\frac{A}{B}\varepsilon\right)^{1/2} + o(|\varepsilon|^{1/2})$, $\tilde{\eta}_0(\varepsilon) = (1, 0, 0, 0)^T \left(-\frac{A}{B}\varepsilon\right)^{1/2} + o(|\varepsilon|^{1/2})$, where $\tilde{\eta}_0(\varepsilon)$ is the solution of reduced BEq ($\beta_1 = \beta_2 = 0$).

$$t_\eta(\tilde{\eta}_0(\varepsilon), \varepsilon) \left[\Lambda_i \tilde{\eta}_0(\varepsilon)\right] = 0, i = 1, 2 \tag{9}$$

where $\Lambda_i$ are infinitesimal operators of Lie algebra in $\Xi_\varphi^4$.

$$t_\eta(\tilde{\eta}_0(\varepsilon), \epsilon) = diag\left\{A\varepsilon - 3A\varepsilon, 0, 0, A\varepsilon - \frac{CA}{B}\varepsilon\right\}.$$

$$\Lambda_1 \tilde{\eta}_0(\varepsilon) = \frac{\partial A_\beta}{\partial \beta_1}|_{\beta_1=\beta_2=0} \cdot \tilde{\eta}_0(\varepsilon) = \frac{\pi}{\sqrt{ab}}(0, 0, -m_1 a, 0)^T \left(-\frac{A}{B}\varepsilon\right)^{1/2},$$

$$\Lambda_2 \tilde{\eta}_0(\varepsilon) = \frac{\partial A_\beta}{\partial \beta_2}|_{\beta_1=\beta_2=0} \cdot \tilde{\eta}_0(\varepsilon) = \frac{\pi}{\sqrt{ab}}(0, -m_1 a, 0, 0)^T \left(-\frac{A}{B}\varepsilon\right)^{1/2},$$

The relations (9) are fulfilled and the stability of branching equations$A_\beta \tilde{\eta}_0(\varepsilon)$ is determined by the signs of the main terms relative to $\varepsilon$ of the eigenvalues of Jacobian matrix $J$ at this solution, which have form ($\tilde{B} = B + C$, $\tilde{C} = 3B - C$) $\nu_{1,2} = 0, \nu_3 = -2A\varepsilon, \nu_4 = \frac{2A\varepsilon}{\tilde{B}+\tilde{C}}(\tilde{C} - \tilde{B})$.

**Theorem 1.** In order to the family of solutions be stable it is necessary and sufficient that the condition $sign\varepsilon = signB = sign(\tilde{B} + \tilde{C}) = -1$ is fulfilled

$$\begin{cases} \tilde{B} + \tilde{C} < 0 \\ \tilde{C} - \tilde{B} > 0 \end{cases} <=> 0 < \frac{|\tilde{C}|}{|\tilde{B}|} < 1 \tag{10}$$

Let us consider the second group of solutions. Here $\nu_{1,2} = -\frac{2AB\varepsilon}{B+C}$, $\nu_3 = -\frac{2A\varepsilon}{B+C}(C - B)$, $\nu_4 = 0$.

**Theorem 2.** In order to the family of solutions be stable it is necessary and sufficient that the condition $sign\varepsilon = sign(B + C) = sign\tilde{B} = -1$ is fulfilled.

$$0 < \frac{|\tilde{B}|}{|\tilde{C}|} < 1 \tag{11}$$

**Remark 1.** At the satisfaction of inequality (10) (resp. (11)), the family (1)-(3) will be stable relative to perturbations of the same periodicity lattices class, and the instability relative to perturbations of the same periodicity class means the instability in general. **Remark 2.** The results for the problem, considered in [6], are similar to the ones obtained for the case of deep fluid (the same conditions for stability, but different expressions for coefficients).

*n=dim N(B)=6.* Here the eigenvalues of Jacobian matrix have form $\nu_{1,2,3,4} = 0$, $\nu_{5,6} = \frac{C\varepsilon \pm \sqrt{C^2\varepsilon^2 + 4AC\varepsilon^2}}{2}$. Since $A, C < 0$, one of them is positive, so we have the instability in this case.

*n=dim N(B)=4+2+2.* For the first solution $\nu_{1,2,3,4,5} = 0$, $\nu_6, \nu_7, \nu_8$ are the roots of cubic equation (one of them is negative and the others are complex-conjucate). At the $\gamma \to 0$ we obtain the conditions under which the solution will be stable: $signBD = -1$, $\frac{EG}{B^2} > 1$. For the second solution $\nu_{1,2,3,4,5,6} = 0$, $\nu_{7,8} = \frac{(F+2C)\varepsilon \pm \sqrt{(F+2C)^2\varepsilon^2 - 8CF(1+\varepsilon^2)}}{2}$. Since $C, F < 0$, for any $\varepsilon > 0$ $\nu_{7,8} < 0$, i.e. the solution is stable. And for the third one: $\nu_{1,2,3,4,5,6} = 0$, $\nu_{7,8} = \frac{F\varepsilon \pm \sqrt{F^2\varepsilon^2 + 8CF\varepsilon^2}}{2}$. Since $C, F < 0$, one of $\nu_{7,8}$ is positive, then the solution is unstable.

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