Towards a solution of the three-dimensional Wagner problem.

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1) Introduction

An algorithm is proposed to solve the three-dimensional Wagner problem [1] for an arbitrary blunt body penetrating initially flat free surface. The flow is described under the classical assumptions of potential flow theory without account for gravity and surface tension. In the continuity of [2], the boundary value problem is formulated in terms of the displacement potential, which is time integral of the velocity potential, and a conformal mapping of the wetted surface onto the unit disk is used. This means that regular enough shapes of entering bodies are only considered. It is shown how to reduce the three-dimensional boundary value problem to a non-linear system of ordinary differential equations, hence yielding a method of solution to this still open problem.

2) Boundary value problem

The boundary value problem (BVP) is formulated in terms of the displacement potential φ

$$\begin{cases} \Delta\varphi = \varphi_{,xx} + \varphi_{,yy} + \varphi_{,zz} = 0 & z < 0 \\ \varphi = 0 & z = 0,\ (x, y) \in \mathrm{FS}(t) \\ \varphi_{,z} = -h(t) + f(x, y) & z = 0,\ (x, y) \in \mathrm{D}(t) \\ \varphi \to 0 & (x^2 + y^2 + z^2) \to \infty, \end{cases} \tag{1}$$

where the regions FS(t) and D(t) are disconnected parts of the plane $z = 0$ and correspond to the free surface and the wetted area of the body, respectively. A closed curve, which separates the regions FS(t) and D(t), is denoted $\Gamma(t)$ and is referred to as the contact line. The body shape is represented by the equation $z = f(x, y)$, where $f(x, y)$ is a smooth positive shape function, and $h(t)$ is the penetration depth of the body into the liquid. The contact line $\Gamma(t)$ in (1) is unknown in advance and its shape can be obtained with the help of the Wagner condition [1], which requires that both the displacement potential $\varphi(x, y, 0, t)$ and the vertical displacements of the liquid particles $\varphi_{,z}(x, y, 0, t)$ of the liquid boundary are continuous at the contact line.

Alternatively one can formulate the boundary value problem of water impact in terms of another variable $V(x, y, z, t)$ which is the vertical displacement of a fluid particle: $V = -\varphi_{,z}$

$$\begin{cases} \Delta V = 0 & z < 0 \\ V = h(t) - f(x, y) & z = 0,\ (x, y) \in \mathrm{D}(t) \\ V_{,z} = 0 & z = 0,\ (x, y) \in \mathrm{FS}(t) \\ V \to 0 & (x^2 + y^2 + z^2) \to \infty. \end{cases} \tag{2}$$

It is shown in [3] that the mixed BVP (2) has a unique solution provided that the RHS of the Dirichlet condition is smooth enough. Consequently it is easy to show that the solution $V(x, y, z, t)$ is induced by a unique distribution of sources over the surface D(t) and V can be expressed as a simple layer potential

$$V(x, y, z, t) = \frac{1}{2\pi} \int_{D(t)} \frac{S(x_0, y_0, t)\ \mathrm{d}x_0 \mathrm{d}y_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}}. \tag{3}$$

It is important to know that $V(x, y, z, t)$ defined by equation (3) is continuous through out the fluid domain $z \leq 0$ including its boundary $z = 0$. In particular, $V(x, y, 0, t)$ is continuous at the contact line $\Gamma(t)$. This means that the Wagner condition, which requires finite and continuous vertical displacement at $\Gamma(t)$ is automatically satisfied through the formulation (3).

Differentiating (3) with respect to z and putting $z = 0$ provide

$$\Delta_2\varphi = S(x, y, t) \qquad \text{for } (x, y) \in \mathrm{D}(t). \tag{4}$$

Substituting (3) into the Dirichlet condition from (2) gives the integral equation for the source distribution $S(x, y, t)$ over $\mathrm{D}(t)$

$$\frac{1}{2\pi}\int_{D(t)} \frac{S(x_0, y_0, t)\,\mathrm{d}x_0\mathrm{d}y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = h(t) - f(x, y). \tag{5}$$

The solution of equation (5) is unique. In [4] the behavior of the solution S close to the contact line $\Gamma(t)$ was analyzed. It was proved that the solution has a square root singularity at the boundary of the region $\mathrm{D}(t)$.

If the contact region $\mathrm{D}(t)$ is known, the integral equation (5) can be solved and the displacement potential in $\mathrm{D}(t)$ can be obtained thereafter as the solution of the Poisson equation (4), which satisfies the following boundary condition

$$\varphi = 0 \qquad \text{on } \Gamma(t). \tag{6}$$

Additional condition is suggested to prescribe along the contact line. This condition implies that not only the displacement potential $\varphi(x, y, 0, t)$ and the vertical displacement $\varphi_{,z}(x, y, 0, t)$ but also the horizontal displacements $\varphi_{,x}(x, y, 0, t)$ and $\varphi_{,y}(x, y, 0, t)$ are continuous through $\Gamma(t)$. With account for (6) the latter condition can be presented as

$$\varphi_{,n} = 0 \qquad \text{on } \Gamma(t) \tag{7}$$

where $\varphi_{,n} = \varphi_{,x}(x, y, 0, t)n_x + \varphi_{,y}(x, y, 0, t)n_y$ and $\mathbf{n} = (n_x, n_y)$ is the unit normal vector along the contact line $\Gamma(t)$. Condition (7) is used below to determine the position of the contact line at each time instant.

3) Methods of solution

Several methods of solution can be proposed to solve the integral equation (5). Some of them are summarized in [6] (pp 201–206). Here we face the difficulty that the contact line is unknown. The operator T defined as

$$T\left[S(x, y, t)\right] = \int_{D(t)} \frac{S(x_0, y_0, t)\mathrm{d}x_0\mathrm{d}y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, \tag{8}$$

has properties for which Hilbert-Schmidt theorem can be applied in spite of the weak singularity of S on the boundary of $\mathrm{D}(t)$. In particular the set of eigenvalues of T is countable and the corresponding eigenfunctions exist. The family of the eigenfunctions is complete and the function S can be projected on this family. However, the eigenfunctions of the operator T cannot be easily obtained in general case and numerical calculations are required. Moreover, it is not clear how the contact line can be found within formulation (4) - (7).

In order to present the formulation (4) - (7) in a form suitable for numerical calculations, we introduce the conformal mapping $g(\omega, t)$. It transforms the wetted surface $\mathrm{D}(t)$, described in the physical plane $Z = x + iy$, into a unit disk C_1 described in the transformed plane $\zeta = \xi + i\eta = \rho e^{i\alpha}$. A possible expression of g is the integer series

$$g(\omega, t) = \sum_{n=1}^{\infty} b_n(t)\omega^n \tag{9}$$

where the complex coefficients depend on time t. Besides they verify $\Re(b_1) \neq 0$ and $\Im(b_1) \equiv 0$. For mathematical manipulation, the coefficients b_n are ranged in a infinite vector $\mathbf{b}(t)$. The convergence of (9) strongly depends on both the smoothness and the general aspect of the contact line $\Gamma(t)$. The smoother $\Gamma(t)$ and the closer aspect ratio to unity, the smaller the required number of terms in the series (9). Other features regarding the regularity of $\Gamma(t)$ must be analyzed. In particular [5] (p 222) analyzed the convexity of $\mathrm{D}(t)$ in terms of the relative importance of the coefficients b_n. We can hence expect that elongated shapes with sharp corners will be the most pathological cases and hence their study will require a large number of terms in the series (9). It is worth noting that the property of g to be conformal means that $dg/d\omega$ cannot vanish on the unit disk. Thus its value at the center of the disk $|\omega| = 0$ can be considered as a length scale of the wetted surface. It is clear that this quantity will play an important role in the convergence of the series.

By introducing the conformal mapping in the integral equation (5), we obtain

$$\frac{1}{2\pi}\int_{C_1} \frac{s(\rho_0, \alpha_0, t)\mathrm{d}\sigma_0}{|g(\omega, t) - g(\omega_0, t)|} = h(t) - f(\Re(g(\omega, t)), \Im(g(\omega, t))), \tag{10}$$

where the elemental surface is denoted $\mathrm{d}\sigma_0 = \rho_0\mathrm{d}\rho_0\mathrm{d}\alpha_0$. Equations (4)-(7) in the new variables take the forms

$$\Delta_2\varphi = s(\rho, \alpha, t) \qquad (\rho < 1), \tag{11}$$

$$\varphi = 0, \quad \partial\varphi/\partial\rho = 0 \qquad (\rho = 1). \tag{12}$$

In general case, a solution of the Poisson equation (11) cannot satisfy both Dirichlet and Neumann conditions (12). It was shown that a solution of the boundary value problem (11), (12) exists if and only if the right-hand side of equation (11) is such that

$$\int_{C_1} \frac{s(\rho_0, \alpha_0, t)(1-\rho_0^2)\mathrm{d}\sigma_0}{1+\rho_0^2 - 2\rho_0\cos[\alpha_0 - \alpha]} = 0 \quad (0 \leq \alpha < 2\pi). \tag{13}$$

Therefore, we can consider first equations (10) and (13), solve them and evaluate the velocity potential as the solution of the boundary value problem (11), (12) at the end. Here equation (13) is used to determine the conformal mapping (9).

The square root singularity of s as $\rho \to 1^-$ suggests to represent this function in the form

$$s(\rho, \alpha, t) = \sum_{m=1}^{\infty} S_m(t) \frac{\Psi_m(\rho, \alpha)}{\sqrt{1-\rho^2}} \tag{14}$$

where the family of functions $\Psi_m(\rho, \alpha)$ is complete on the unit circle.

Now we can formulate a differential system for the unknown variables of the problem, namely the coefficients S_m (ranged in the vector $\mathbf{S}$) and the conformal mapping coefficients $\mathbf{b}$. The differential system is obtained by differentiation of equations (10) and (13) with respect to time t and the projection of the resulting equations on the families $\mathbf{\Psi}(\rho, \alpha)$ and $\mathbf{\Psi}(1, \alpha)$, respectively.

$$\begin{cases} \mathbf{G}(\mathbf{b})\dfrac{d\mathbf{S}}{dt} = \mathbf{A}(\mathbf{b}, \mathbf{S}, t) \\ \dfrac{d\mathbf{b}}{dt} = \mathbf{B}(\mathbf{b}, \mathbf{S}, t) \end{cases} \tag{15}$$

The vectors $\mathbf{A}$ and $\mathbf{B}$ are known functions of $\mathbf{b}, \mathbf{S}, t$. For theoretical reasons mentioned above, the symmetric matrix $\mathbf{G}$ is invertible. A coefficient of this matrix reads

$$G_{nm}(\mathbf{b}) = \frac{1}{2\pi}\int_{C_1} \frac{\Psi_m(\rho, \alpha)}{\sqrt{1-\rho^2}} \left[\frac{1}{2\pi}\int_{C_1} \frac{\Psi_n(\rho_0, \alpha_0)}{\sqrt{1-\rho_0^2}} \frac{\mathrm{d}\sigma_0}{|g(\omega, \mathbf{b}) - g(\omega_0, \mathbf{b})|}\right] \mathrm{d}\sigma \tag{16}$$

This means that the best choice for the functions Ψ_m is such that the matrix $\mathbf{G}$ is diagonal. However it does not seem possible to calculate the family $\mathbf{\Psi}(\rho, \alpha)$ since the conformal mapping function g is a part of the solution. The analysis of the Green function is hence necessary. To this end, we can provide interesting features of following function

$$P(\omega, \omega_0, t) = \left|\frac{\omega - \omega_0}{g(\omega, t) - g(\omega_0, t)}\right| \tag{17}$$

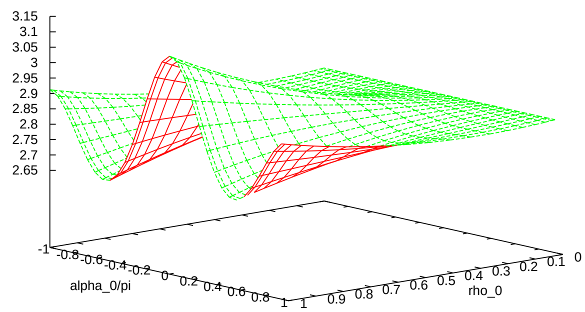
First of all, P is perfectly regular on C_1; for the limiting case $\omega \to \omega_0$, $1/P$ tends to $|dg/d\omega|$ which cannot vanish since g is conformal. Moreover, P is symmetric since ω and ω_0 can permute. Last but not least, by using the series (9), P can be broken down as

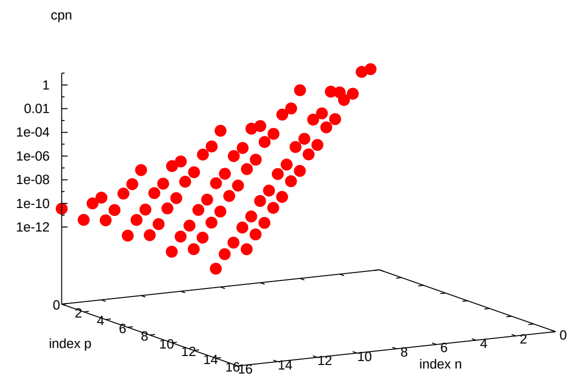
$$P = \sum_{n=0}^{\infty} \rho_0^n \sum_{p=0}^{n} c_{(p,n)}(\mathbf{b}, \rho, \alpha) e^{i(n-2p)\alpha_0} \tag{18}$$

where the coefficients $c_{(p,n)}$ are easily obtained by recursion for a given vector $\mathbf{b}$ and polar coordinates (ρ, α). It is worth noticing that the exponent of ρ_0 and the corresponding Fourier development $e^{i\alpha_0}$ have the same parity. More the truncation of the Fourier series corresponds to the exponent of the radial coordinate and similar expansion holds for $c_{(p,n)}$ in terms of (ρ, α).

However, an additional difficulty arises when performing the integration in α_0 since the integrand in equation (16) is not continuous at $\alpha = \alpha_0$. In fact, the integrals in α_0 must be calculated as Copson's integrals and then the developments can be further pursued. This means that $\mathbf{\Psi}(\rho, \alpha)$ must be broken down in Fourier series while, in the radial direction, one way is to break down $\mathbf{\Psi}(\rho, \alpha)$ as local polynomials of low order, in the spirit of a Spline interpolation. Another way is to find a suitable decomposition of S valid thoughout the unit disk.

At that stage, we cannot define precisely what is the computational effort to arrive at a converged solution. It is definitely expected that the present algorithm will "cost" much less than numerical approaches like in [8]. There is already one difficulty which has been fully studied. It concerns the convergence of function P broken down in series (18). The following figures illustrate its convergence for a slightly elliptic contact line as defined in [2]. In that case the coefficients b_1 and b_3 are non trivially zero. Function P is then calculated for a given complex number $\omega = \rho e^{i\alpha}$ while the polar coordinates cover the intervals $\rho_0 \in [0 : 1]$ and $\alpha_0 \in [-\pi : \pi]$. The left figure shows the variation of P with (ρ_0, α_0) for which 16 Fourier modes suffice. The right plot illustrates the variation of $c_{(p,n)}$ with the indices (p, n). In this example, the coefficients decrease exponentially as $|c_{(p,n)}| \sim e^{-3n/2}$.





5) Conclusion

Under the assumptions that we can map the wetted surface onto a unit circle, it is shown that the three-dimensional Wagner problem can be formulated as a time differential system for the planar Laplacian of the displacement potential and for the conformal mapping function. Some mathematical properties of the corresponding differential system were established. Alternatively, but maybe of lesser interest, a nonlinear system of equations for the mapping function can be formulated. Numerical applications are now performing to test the stability of the proposed algorithms with respect to known solutions of problem (see [7]).

6) References

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