Time-dependent interaction of water waves and a vertical elastic plate

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1 Introduction

We consider the problem of a vertical elastic plate which forms the boundary of a semi-infinite two-dimensional region of fluid. This is considered as a simple model of wave interaction with elastic-walled tanks. We calculate the solution in the time domain by an expansion in the solutions for a single frequency. The single-frequency solution is found from a double eigenfunction expansion in the eigenfunctions of the elastic plate and the eigenfunctions for water of constant depth. The solution in the time domain is then written in an expansion over the single-frequency solutions using a generalised eigenfunction expansion. This method requires a special inner product in which the linear evolution operator in the time domain is unitary. The method differs substantially from other time-dependent methods such as time stepping or the so called memory-effect method. The generalised eigenfunction method goes back to the work of Povzner (1953) and has been recently used in the context of water waves by Hazard & Lenoir (2002); Hazard & Loret (2007). The problem and method presented here is similar to that developed by Hazard & Meylan (2008) for the floating elastic plate, except for a modification required by the fact that the present problem is semi-infinite, as well as for the obvious difference in formulation. As well, the proof of self-adjointness is more complicated in the situation considered here. It is worth noting that all integrals arising in the solution of the single-frequency case have been calculated analytically so no numerical integration is required.

2 Problem formulation

The problem consists of a semi-infinite domain $-h < z < 0$ and $-\infty < x < 0$, where $z = 0$ represents the free surface Γ^{f} and where there is an elastic plate Γ^{p} at $x = 0$ extending from $-h$ to H with $H \geq 0$. The plate is assumed to be governed by the Bernoulli–Euler plate equation subject to fixed boundary conditions at the bottom and fixed or free conditions at the top. Thus, we have the infinite set of modes $w_n(z)$ which satisfy the equation $\partial_z^4 w_n = \lambda_n^4 w_n$ and the boundary conditions at the ends of the plates, which we discuss in §4. This gives us the following expression for the displacement of the plate $W(z,t)$:

$$W(z,t) = \sum_{n=0}^{\infty} \alpha_n(t) w_n(z).$$

2.1 Governing equations

We consider the linear problem in non-dimensional form, where the spatial variables have been scaled with respect to a length parameter L and the time variable with respect to $\sqrt{L/g}$, where g is the acceleration due to gravity.

The mathematical description of the problem is as follows. We use the water *acceleration* potential $\Psi(x,z,t)$, which satisfies

$$-\Delta\Psi = 0, \qquad -h < z < 0, \tag{1}$$

$$\partial_n \Psi = 0, \qquad z = -h, \tag{2}$$

where n is the outward unit normal. The kinematic condition at the free surface is

$$\partial_t^2 \zeta = \partial_n \Psi, \quad z = 0, \tag{3}$$

where $\zeta(x,t)$ is the displacement of the water surface. The dynamic condition is

$$\zeta + \Psi = 0, \quad z = 0. \tag{4}$$

The plate is governed by the plate equation. There is a force on the wetted surface of the plate from the water (given by $p = -\Psi$) while there is no force above the free water surface. The governing equation thus reads

$$\gamma \partial_t^2 W + \beta \partial_z^4 W = \begin{cases} 0, & z > 0, \\ -\Psi, & z \leq 0, \end{cases} \tag{5}$$

subject to edge conditions discussed in §4. For convenience, we write χ for the characteristic function of the wetted plate, i.e. $\chi(z) = 1$ for $z < 0$ and $\chi(z) = 0$ for $z \geq 0$.

At $x = 0$ we have to match the plate displacement with the pressure and with the kinematic condition. The equations coupling the water acceleration potential and the plate displacement are therefore

$$-\chi\Psi = \gamma \sum_{n=0}^{\infty} \partial_t^2 \alpha_n w_n + \beta \sum_{n=0}^{\infty} \lambda_n^4 \alpha_n w_n \qquad (6)$$

and, along the wetted plate surface,

$$\sum_{n=0}^{\infty} \partial_t^2 \alpha_n w_n = \partial_n \Psi. \qquad (7)$$

For $z < 0$, we combine the last two equations to give

$$-\Psi - \gamma \partial_n \Psi = \beta \sum_{n=0}^{\infty} \lambda_n^4 \alpha_n w_n.$$

The system is completed by initial conditions at $t = 0$ prescribing free surface and plate displacement and velocity.

3 Harmonic time dependence

We assume that the acceleration potential and the surface displacement are time harmonic with radian frequency ω. E.g. we have $\Psi(x, z, t) = \mathbb{R}e\,\{\psi(x, z)\mathrm{e}^{-\mathrm{i}\omega t}\}$ with a complex potential ψ.

At the free surface, condition (3) simplifies and we can combine it with equation (4) to give the single free-surface condition

$$\alpha\psi = \partial_z \psi, \quad z = 0, \qquad (8)$$

where $\alpha = \omega^2$.

The acceleration potential takes the form

$$\psi(x, z) = A f_0(z) \mathrm{e}^{-k_0 x} + \sum_{m=0}^{\infty} c_m f_m(z) \mathrm{e}^{k_m x}, \qquad (9)$$

where the first term is due to the ambient incident potential of amplitude A and the coefficients c_m are of the scattered wavefield only. The functions

$$f_m(z) = \frac{\cos k_m(z + h)}{\cos k_m h}$$

are the vertical eigenfunctions. The numbers k_m, $m \geq 1$, are given as positive real roots of the dispersion relation

$$\alpha + k_m \tan k_m h = 0. \qquad (10)$$

The positive wavenumber k is related to α by the dispersion relation

$$\alpha = k \tanh kh. \qquad (11)$$

For ease of notation, we write $k_0 = -\mathrm{i}k$. Note that k_0 is a (purely imaginary) root of (10). Moreover, we have

$$\int_{-h}^{0} f_m(z) f_n(z)\, \mathrm{d}z = \delta_{mn} N_m,$$

where N_m is given by

$$N_m = \frac{1}{2} \frac{\cos k_m h \sin k_m h + k_m h}{k_m \cos^2 k_m h}.$$

Using (9) at $x = 0$ in (6) and making use of the orthogonality of the w_k we arrive at

$$- A \int_{-h}^{0} w_k f_0\, \mathrm{d}z - \sum_{m=0}^{\infty} c_m \int_{-h}^{0} w_k f_m\, \mathrm{d}z = (-\gamma\omega^2 + \beta\lambda_k^4)\alpha_k. \qquad (12)$$

The second equation can be obtained in the same way from (7),

$$- \omega^2 \sum_{n=0}^{\infty} \alpha_n \int_{-h}^{0} w_n f_l\, \mathrm{d}z = -k_0 N_0 A \delta_{l0} + k_l N_l c_l. \qquad (13)$$

Solving (13) for c_l and substituting into (12), the coefficients c_l can be eliminated. Simple manipulations lead to

$$\sum_{m=0}^{\infty} \frac{\omega^2}{k_m N_m} \sum_{n=0}^{\infty} \alpha_n \int_{-h}^{0} w_n f_m\, \mathrm{d}z \int_{-h}^{0} w_k f_m\, \mathrm{d}z = (-\gamma\omega^2 + \beta\lambda_k^4)\alpha_k + 2A \int_{-h}^{0} w_k f_0\, \mathrm{d}z \qquad (14)$$

as the determining system of equations for the unknown coefficients α_k. The coefficients c_l can easily be calculated from (13) once the α_k are known.

4 The eigenfunctions of the plate

The eigenfunctions of the bi-harmonic operator subject to fixed boundary conditions at the bottom of the plate, i.e.

$$w_k(-h) = 0, \quad w_k'(-h) = 0, \qquad (15)$$

are generally given as

$$w_k(z) = C_2(\cos \lambda_k(z + h) - \cosh \lambda_k(z + h)) + C_4(\sin \lambda_k(z + h) - \sinh \lambda_k(z + h)). \qquad (16)$$

4.1 Fixed plate top

If the plate is also fixed at $z = H$, we have

$$w_k(H) = 0, \quad w_k'(H) = 0$$

in addition to (15). Using this in (16), we obtain

$$C_4 = C_2 \frac{\sin \lambda_k(H+h) + \sinh \lambda_k(H+h)}{\cos \lambda_k(H+h) - \cosh \lambda_k(H+h)}.$$

We choose C_2 such that w_k has unit norm.

The determining equation for the eigenvalues turns out to be given by

$$\cos \lambda_k(H+h) \cosh \lambda_k(H+h) = 1.$$

4.2 Free plate top

If the plate is free to move at $z = H$, we have

$$w_k''(H) = 0, \quad w_k'''(H) = 0$$

in addition to (15). Hence,

$$C_4 = C_2 \frac{\sin \lambda_k(H+h) - \sinh \lambda_k(H+h)}{\cos \lambda_k(H+h) + \cosh \lambda_k(H+h)}.$$

Again, C_2 is chosen such that w_k has unit norm.

In this case, the determining equation for the eigenvalues is found to be given by

$$\cos \lambda_k(H+h) \cosh \lambda_k(H+h) = -1.$$

5 Time-dependent problem

We want to use the time-harmonic solutions for fixed frequencies to construct the solution for arbitrary given initial surface and plate displacement and velocity. This is quite simple if the plate is at rest initially and the incident wave is far away from the plate (cf. §5.1). For the general case, we develop a spectral theory in §5.2.

5.1 Far away incident wave

Assuming that the initial surface velocity is zero and the surface displacement is given by $f(x)$, we calculate its wavenumber components by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \mathrm{e}^{-\mathrm{i}kx} \,\mathrm{d}x.$$

If the wave is sufficiently far away from the plate initially, the time-dependent water acceleration potential Ψ is given by the Fourier-type integral,

$$\Psi(x,z,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(k) \hat{\psi}(x,z,k) \mathrm{e}^{-\mathrm{i}\omega(k)t} \,\mathrm{d}k, \tag{17}$$

where $\hat{\psi}(x,z,k)$ is the acceleration potential calculated for a time-harmonic incident wave of unit amplitude and wavenumber k. Note that the radian frequency ω is related to the wavenumber k by the dispersion relation (11). The surface displacement $\zeta(x,t)$ can be calculated from the potential via relation (4). Since $\hat{\psi}(x,z,-k) = \bar{\hat{\psi}}(x,z,k)$, expression (17) can be simplified,

$$\Psi(x,z,t) = \frac{2}{\sqrt{2\pi}} \mathbb{R}e \left[\int_0^\infty \hat{f}(k) \hat{\psi}(x,z,k) \mathrm{e}^{-\mathrm{i}\omega(k)t} \,\mathrm{d}k \right].$$

5.2 Spectral formulation

We want to write the equations of motion in the form of an abstract wave equation. We introduce a two component system

$$\partial_t^2 \begin{pmatrix} \zeta \\ W \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} \zeta \\ W \end{pmatrix}.$$

The operators $\mathcal{A}_{ij}$ are mapping in the following way: $\mathcal{A}_{11} : \zeta \mapsto \Psi_n|_{\Gamma^\mathrm{f}}$, $\mathcal{A}_{12} : W \mapsto \Psi_n|_{\Gamma^\mathrm{f}}$, $\mathcal{A}_{21} : \zeta \mapsto \partial_t^2 W|_{\Gamma^\mathrm{p}}$ and $\mathcal{A}_{22} : W \mapsto \partial_t^2 W|_{\Gamma^\mathrm{p}}$ In all cases, Ψ is the solution of the boundary-value problem

$$\begin{aligned} -\Delta\Psi &= 0, & -h < z < 0, \\ \partial_z \Psi &= 0, & z = -h, \end{aligned}$$

and the boundary conditions are given by:

	free surface	wetted plate
$\mathcal{A}_{j1}$	$\Psi = -\zeta$	$\Psi = -\gamma\Psi_n$
$\mathcal{A}_{j2}$	$\Psi = 0$	$\Psi = -\gamma\Psi_n - \beta\partial_z^4 W$

For A_{2j} the potential at the wetted plate further needs to be mapped to the acceleration of the full plate. For $\mathcal{A}_{21}$, we have to use

$$\gamma \partial_t^2 W = -\chi\Psi,$$

while for $\mathcal{A}_{22}$, we require

$$\gamma \partial_t^2 W = -\beta\partial_z^4 W - \chi\Psi,$$

each subject to the edge conditions.

The operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}$$

is self-adjoint in its domain equipped with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ W \end{pmatrix}, \begin{pmatrix} \zeta' \\ W' \end{pmatrix} \right\rangle_{\mathcal{A}} = \langle \zeta, \zeta' \rangle_{\Gamma^\mathrm{f}} + \beta \langle \partial_z^2 W, \partial_z^2 W' \rangle_{\Gamma^\mathrm{p}}.$$

It has a continuous spectrum: $(-\infty, 0]$. The corresponding generalised eigenfunctions are just the single-frequency solutions $\big(\hat{\zeta}(x,k), \hat{w}(z,k)\big)$.

We have

$$\left\langle \begin{pmatrix} \hat{\zeta}(\,\cdot\,,k) \\ \hat{w}(\,\cdot\,,k) \end{pmatrix}, \begin{pmatrix} \hat{\zeta}(\,\cdot\,,\kappa) \\ \hat{w}(\,\cdot\,,\kappa) \end{pmatrix} \right\rangle_{\mathcal{A}} = 2\pi\delta(k-\kappa).$$

We also know that

$$\begin{pmatrix} \zeta(x,t) \\ W(z,t) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int_0^\infty f_1(k) \begin{pmatrix} \hat{\zeta}(x,k) \\ \hat{w}(z,k) \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega(k)t} + f_2(k) \begin{pmatrix} \hat{\zeta}(x,k) \\ \hat{w}(z,k) \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega(k)t}\,\mathrm{d}k. \quad (18)$$

Thus, denoting the initial displacement of the free surface and the plate by ζ_0 and w_0, resp., we obtain

$$\begin{pmatrix} \zeta_0(x) \\ w_0(z) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int_0^\infty f_1(k) \begin{pmatrix} \hat{\zeta}(x,k) \\ \hat{w}(z,k) \end{pmatrix} + f_2(k) \begin{pmatrix} \hat{\zeta}(x,k) \\ \hat{w}(z,k) \end{pmatrix} \mathrm{d}k$$

and we find that

$$\langle \zeta_0, \hat{\zeta}(\,\cdot\,,\kappa)\rangle_{\Gamma^{\mathrm{f}}} + \beta \langle \partial_z^2 w_0, \partial_z^2 \hat{w}(\,\cdot\,,\kappa)\rangle_{\Gamma^{\mathrm{p}}} = \left\langle \begin{pmatrix} \zeta_0 \\ w_0 \end{pmatrix}, \begin{pmatrix} \hat{\zeta}(\,\cdot\,,\kappa) \\ \hat{w}(\,\cdot,\,\kappa) \end{pmatrix} \right\rangle_{\mathcal{A}} = \sqrt{2\pi}(f_1(\kappa)+f_2(\kappa)).$$

Hence,

$$\int_{\Gamma^{\mathrm{f}}} \zeta_0(x)\bar{\hat{\zeta}}(x,\kappa)\,\mathrm{d}x + \beta \int_{\Gamma^{\mathrm{p}}} \partial_z^2 w_0(z) \partial_z^2 \bar{\hat{w}}(z,\kappa)\,\mathrm{d}z = \sqrt{2\pi}(f_1(\kappa)+f_2(\kappa)).$$

and, similarly,

$$\int_{\Gamma^{\mathrm{f}}} \zeta_0'(x)\bar{\hat{\zeta}}(x,\kappa)\,\mathrm{d}x + \beta \int_{\Gamma^{\mathrm{p}}} \partial_z^2 w_0'(z) \partial_z^2 \bar{\hat{w}}(z,\kappa)\,\mathrm{d}z = \mathrm{i}\sqrt{2\pi}\omega(\kappa)(-f_1(\kappa)+f_2(\kappa)).$$

are obtained, which determine the functions f_1 and f_2 in the representation (18).

6 An example

Consider the problem where the free surface is initially at rest and the plate is bent, i.e. w_0 is non-zero and $\zeta_0' = 0$, $w_0' = 0$ and $\zeta_0 = 0$. This gives $f_1 = f_2 = f$ say, and

$$f(\kappa) = \frac{1}{2\sqrt{2\pi}} \beta \int_{\Gamma^{\mathrm{p}}} \partial_z^2 w_0(z) \partial_z^2 \bar{\hat{w}}(z,\kappa)\,\mathrm{d}z.$$

For $w_0(z) = \sum_{n=0}^\infty \alpha_n^0 w_n(z)$, we have

$$f(\kappa) = \frac{1}{2\sqrt{2\pi}} \beta \sum_{n=0}^\infty \alpha_n^0 \lambda_n^4 \bar{\hat{\alpha}}(\kappa). \quad (19)$$

We choose $d = 1$, $H = 0$, $\beta = \gamma = 0.01$ and $\alpha_n^0 = -0.5\,\delta_{0n}$ and present results for the fixed and free plate top in figures 1 and 2, resp. It can be observed that the free plate creates a wave travelling away from the plate but the system comes to rest quickly in the vicinity of the plate. For the fixed plate, however, a mode is excited which is sustained for a long time. This can also be seen as a sharp peak at the corresponding wavenumber in the function $f(\kappa)$ given by (19). A closer investigation of this phenomenon is currently being undertaken.

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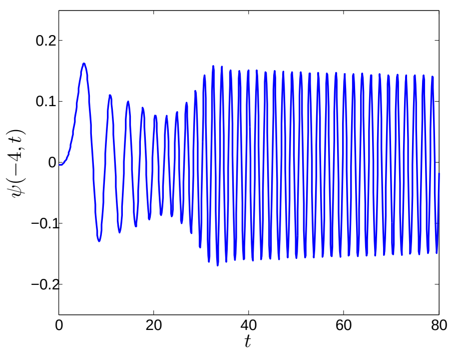

Fig. 1: Time evolution of the free surface displacement at $x = -4$ for plate with fixed top.

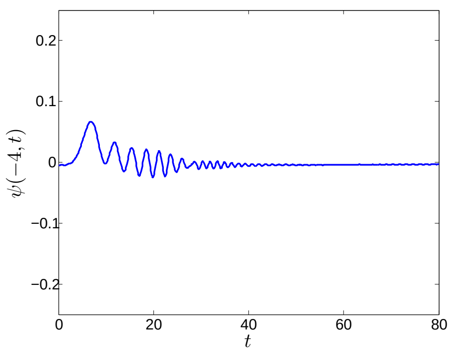

Fig. 2: Time evolution of the free surface displacement at $x = -4$ for plate with free top.