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# Asymptotic solutions for flexural-gravity waves due to a transient disturbance

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The classical Cauchy-Poisson water wave problems (CPWWP) are closely related with the wave motions due to disturbances originating at the free surface. There have been several attempts to extend the CPWWP to complicated physical configurations, including the effects of the viscosity [1, 2], inertial surface [3], and surface tension [4]. Lu & Chwang [5] introduced the laminar interaction of a Stokeslet with a free surface. Chen *et al.* [6, 7] considered the combined effects of the surface tension and viscosity. Lu *et al.* [8] studied the cases of two semi-infinite inviscid fluids. Recently, Maiti & Mandal [9] and Lu & Dai [10] performed an asymptotic analysis on the flexural-gravity waves due to impulsive disturbance exerted on the surface of an inviscid fluid covered by an infinite ice sheet.

In this paper, the dynamic responses of an ice-covered fluid to a submerged point mass source are further investigated. The initially quiescent fluid of infinite depth is assumed to be inviscid, incompressible and homogenous. The thin ice-cover is modeled as a homogenous elastic plate with negligible inertia. It is assumed that the wave amplitudes generated are very small in comparison with the wavelength. The linearized initial-boundary-value problem is formulated within the framework of potential flow theory. The solutions in integral form for the vertical deflexions at the ice-water interface are obtained by means of a joint Laplace-Fourier transform. The asymptotic representations of the wave motions for large time with a fixed distance-to-time ratio are derived by making use of the generalized method of stationary phase. Two special cases of the problem considered herein are derived, corresponding to the gravity waves on an inertial and clean free surface.

The Cartesian coordinates are used in which the $z$ axis points vertically upward while $z = 0$ represents the mean ice-water interface. The governing equation is

$$\nabla^2\Phi = M\delta(\mathbf{x}-\mathbf{x}_0)\delta(t-t_0), \tag{1}$$

where $\Phi(\mathbf{x}, t; \mathbf{x}_0, t_0)$ is the velocity potential for the perturbed flow, $M$ the constant strength of the simple source, $\delta(\cdot)$ the Dirac delta function, $\mathbf{x} = (x, y, z)$ an observation point, $t$ the time, $\mathbf{x}_0 = (0, 0, z_0)$ the source point and $t_0$ the instant at which the source is applied. Without loss of generality, we set $t_0 = 0^+$. The linearized boundary conditions will be applied on the undisturbed ice-water interface. The kinematic and dynamic conditions at $z = 0$ are given by

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\Phi}{\partial z} = 0, \tag{2}$$

$$\rho\frac{\partial\Phi}{\partial t} + \rho g\zeta + D\nabla^4\zeta + \rho_e h\frac{\partial^2\zeta}{\partial t^2} = 0, \tag{3}$$

where $\zeta$ is the vertical deflexion of the ice-water interface; $\rho$ and $\rho_e$ are the uniform densities of the fluid and the plate, respectively; $g$ is the acceleration of gravity; $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity of the plate; $E$, $h$ and $\nu$ are Young's modulus, the thickness and Poisson's ratio of the plate, respectively. The initial conditions at $z = 0$ are

$$\Phi|_{t=0} = 0, \quad \zeta|_{t=0} = 0, \quad \frac{\partial\zeta}{\partial t}|_{t=0} = 0. \tag{4}$$

Moreover, it is required that $\nabla\Phi \to 0$ as $z \to -\infty$.

We write

$$\Phi = \Phi^S(\mathbf{x}, t; \mathbf{x}_0, t_0) + \Phi^R(\mathbf{x}, t), \tag{5}$$

where $\Phi^S = -M\delta(t)/(4\pi r)$ is the potential due to the simple source in an unbounded domain while $\Phi^R$ is a continuous function everywhere in the corresponding domain and $\nabla^2\Phi^R = 0$, $r = ||\mathbf{x}-\mathbf{x}_0||$. Thus, the relation between the singular and regular components can be established through the boundary conditions at $z = 0$:

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\Phi^R}{\partial z} = \frac{\partial\Phi^S}{\partial z}, \tag{6}$$

$$\frac{\partial\Phi^R}{\partial t} + g\zeta + \gamma g\nabla^4\zeta + \sigma\frac{\partial^2\zeta}{\partial t^2} = -\frac{\partial\Phi^S}{\partial t}, \tag{7}$$

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where $\gamma = D/\rho g$, $\sigma = h\rho_e/\rho$.

It is convenient to introduce a combination of the Laplace transform with respect to $t$ and the Fourier transform with respect to spatial variables:

$$\{\Phi^R, \zeta\} = \frac{1}{8\pi^3 \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}\alpha \int_{-\infty}^{+\infty} \mathrm{d}\beta \times \{\tilde{\Phi}^R \exp(kz), \tilde{\zeta}\} \exp(\mathrm{i}f + st), \quad (8)$$

where $k = \sqrt{\alpha^2 + \beta^2}$, $f = \alpha x + \beta y$. By substituting Eq. (8) into the Laplace-Fourier transforms of boundary conditions (6) and (7), two simultaneous algebraic equations are set up for the unknown functions $\tilde{\Phi}^R$ and $\tilde{\zeta}$, which can readily be solved. Consequently, the formal integral expression for the displacement of ice-water interface can be written as

$$\zeta = \frac{M}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A \cos(\omega t) \exp(\mathrm{i}f) \mathrm{d}\alpha \mathrm{d}\beta, \quad (9)$$

where

$$A(k, h, z_0) = \frac{\exp(kz_0)}{1 + \sigma k}, \quad (10)$$

$$\omega(k, \gamma, \sigma) = \left(\frac{1 + \gamma k^4}{1 + \sigma k}\right)^{1/2} \sqrt{gk}. \quad (11)$$

With a change of variables

$$\{x, y\} = R\{\cos\theta, \sin\theta\},\ \{\alpha, \beta\} = k\{\cos\phi, \sin\phi\},$$

Equation (9) can be re-written as

$$\zeta = \frac{1}{2\pi} \int_0^{+\infty} Ak \mathrm{J}_0(kR) \cos(\omega t) \mathrm{d}k, \quad (12)$$

where $\mathrm{J}_0(kR)$ is the zeroth-order Bessel function of the first kind. Furthermore, we may replace $\mathrm{J}_0(kR)$ by its asymptotic formula for large $kR$ [11],

$$\mathrm{J}_0(kR) \simeq \left(\frac{2}{\pi kR}\right)^{1/2} \cos\left(kR - \frac{\pi}{4}\right). \quad (13)$$

Thus, we have an approximation for Eq. (12) as follows

$$\zeta \simeq \frac{M}{4\pi} \sum_{m=1}^{2} \sum_{n=1}^{2} \int_0^{+\infty} \mathrm{d}k \times \left(\frac{k}{2\pi R}\right)^{1/2} A \exp(\mathrm{i}t\Theta_{mn}), \quad (14)$$

where

$$\Theta_{mn} = \frac{(-1)^{m+1}}{t}\left(kR - \frac{\pi}{4}\right) + (-1)^{n+1}\omega. \quad (15)$$

Next, the asymptotic behavior of Eq. (14) shall be studied for large $t$ with a fixed distance-to-time ratio by means of the method of stationary phase. According to the stationary-phase approximation, the dominant contribution to the integral in Eq. (14) stems from the stationary points of $\Theta_{mn}$. It is easily seen that $\Theta_{12}$ and $\Theta_{21}$ have stationary points. The solutions for the stationary points, denoted by $k_j$, are determined by

$$\frac{\partial \Theta_{mn}}{\partial k} = 0. \quad (16)$$

A straightforward derivation for Eq. (16) yields

$$Q(k, \gamma, \sigma) = \frac{R}{t} - C_g = \frac{\sqrt{g}}{2}\left[\frac{1}{\sqrt{k_0}} - \frac{1}{\sqrt{k}} \cdot \frac{1 + 5\gamma k^4 + 4\gamma\sigma k^5}{(1 + \sigma k)^{3/2}(1 + \gamma k^4)^{1/2}}\right] = 0, \quad (17)$$

where $C_g(k, h) = \partial\omega/\partial k$ is the group velocity, and $k_0 = gt^2/4R^2$. It is noted that $k_0$ is the special root of $Q(k, 0, 0) = 0$, which corresponds to the Cauchy-Poisson gravity waves.

Generally speaking, the explicit analytical solutions for Eq. (17) cannot readily be given for arbitrary $h$, $\gamma$, $\sigma$, and $R/t$. However, once the physical parameters $h$, $\gamma$ and $\sigma$ are given, the nature of roots with respect to $k$ for Eq. (17) depends on the value of $R/t$ only. To have a graphical representation for the theoretical results, we adopt hereinafter physical parameters given by Squire *et al.* [12], $E = 5$GPa, $\nu = 0.3$, $\rho = 1024\mathrm{kgm}^{-3}$, $\rho_e = 917\mathrm{kgm}^{-3}$ and $g = 9.8\mathrm{ms}^{-2}$. Figure 1 shows curves for the group velocities $C_g(k, h)$. It can be seen from Fig. 1 that for a given $h$, there exists a minimal group velocity, denoted by $C_{g\min}(h) = C_g(k_c, h)$, at which Eq. (17) has one real positive root $k_c(h)$ only and $\omega_c'' = \partial^2\omega(k_c, \gamma, \sigma)/\partial k^2 = 0$. As $h$ increases, $C_{g\min}$ increases while $k_c$ decreases. When $R/t > C_{g\min}$, Eq. (17) has two real positive roots, $k_1(R/t, h)$ and $k_2(R/t, h)$ with $0 < k_1 < k_2 < +\infty$. Genernally, $k_1$ and $k_2$ can be numerically obtained from Eq. (17).

When $R/t > C_{g\min}$, according to the standard stationary-phase approximation, the expansion for the phase function near $k_j$ is taken as

$$\Theta_{mn}(k) \approx \Theta_{mn}(k_j) + \frac{1}{2}\frac{\partial^2\Theta_{mn}(k_j)}{\partial k^2}(k - k_j)^2. \quad (18)$$

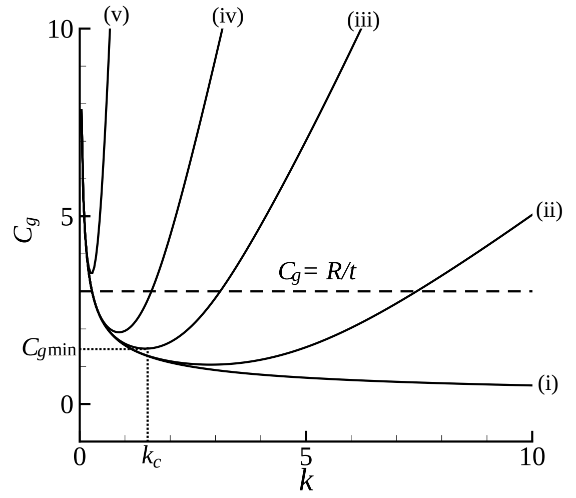


FIG. 1: Group velocity curves $C_g(k,h)$ with (i) $h = 0$m, (ii) $h = 0.002$m, (iii) $h = 0.005$m, (iv) $h = 0.01$m, and (v) $h = 0.05$m.

By a straightforward application of the method of stationary phase, the asymptotic representation of Eq. (14) can be given as

$$\zeta \simeq M \sum_{j=1}^{2} \frac{k_j^{1/2} \exp(k_j z_0) \cos\varphi_j}{2\pi(R|\omega_j''|t)^{1/2}(1+\sigma k_j)}, \quad (19)$$

where

$$\varphi_j = k_j R - \omega_j t - [1 + \mathrm{sgn}(\omega_j'')]\frac{\pi}{4}, \quad (20)$$

$\omega_j = \omega(k_j, \gamma, \sigma)$, and $\omega_j'' = \partial^2\omega(k_j,\gamma,\sigma)/\partial k^2$, $\mathrm{sgn}(x) = \pm 1$ as $x \gtrless 0$. It should be noted that Eq. (19) holds for $\omega_j'' \neq 0$ only.

As $R/t$ decreases to $C_{g\min}$, $k_1$ and $k_2$ move together toward $k_c$ while $\omega_j''$ tends to zero. Accordingly, Eq. (19) predicts an infinitely increasing wave amplitude. It is noted that $\omega_c''' = \partial^3\omega(k_c,\gamma,\sigma)/\partial k^3 \neq 0$. In this case, the expansion for the phase function near $k_c$ is taken as

$$\Theta_{mn}(k) \approx \Theta_{mn}(k_c) + \frac{\partial\Theta_{mn}(k_c)}{\partial k}(k-k_c) + \frac{1}{6}\frac{\partial^3\Theta_{mn}(k_c)}{\partial k^3}(k-k_c)^3. \quad (21)$$

Thus, Eq. (14) can be approximately given as

$$\zeta \simeq M \mathrm{Ai}(Z_c)\left(\frac{2}{|\omega_c'''|t}\right)^{1/3} \times \frac{k_c^{1/2}\exp(k_c z_0)\cos\varphi_c}{(2\pi R)^{1/2}(1+\sigma k_c)}, \quad (22)$$

where

$$Z_c = (\omega_c' t - R)\left(\frac{2}{\omega_c''' t}\right)^{1/3}, \quad (23)$$

$$\varphi_c = k_c R - \omega_c t - \pi/4, \quad (24)$$

and $\omega_c' = \partial\omega(k_c,\gamma,\sigma)/\partial k$, $\mathrm{Ai}(Z)$ is the Airy integral.

A special case of Eqs. (1)-(4) with $\gamma = 0$ and $\sigma \neq 0$ corresponds to the problem for the transient waves due to a submerged point source submerged in an inviscid fluid with an inertial surface. The inertial surface represents the effect of a thin uniform distribution of non-interacting floating matter [3]. It noted that $\partial^2\omega(k,0,\sigma)/\partial k^2 < 0$ holds for all $k > 0$ and $\sigma \neq 0$. In this case, Eq. (17) can be transformed into a biquadratic equation, for which the single real positive root can be exactly given as

$$k_\sigma = \frac{1}{2\sigma}\left\{-\frac{3}{2} + \left(\frac{1}{4} - a\right)^{1/2} + \left[\frac{1}{2} + a + \frac{1}{4}\left(\frac{1}{4} - a\right)^{-1/2}\right]^{1/2}\right\}, \quad (25)$$

$$a = 4\left(\frac{2\sigma^2 k_0^2}{3b}\right)^{1/3} - \left(\frac{\sigma k_0 b}{18}\right)^{1/3}, \quad (26)$$

$$b = -9 + (81 + 768\sigma k_0)^{1/2}. \quad (27)$$

For a fixed $R/t$, $k_\sigma$ decreases from $k_0$ with increasing $\sigma$. As $\gamma = 0$ and $\sigma \neq 0$, Eq. (19) simply reduces to

$$\zeta \simeq \frac{M k_\sigma \exp(k_\sigma z_0)\cos\varphi_\sigma}{2\pi(|\omega_\sigma''|Rt)^{1/2}(1+\sigma k_\sigma)}, \quad (28)$$

where

$$\varphi_\sigma = k_\sigma R - \omega_\sigma t, \quad (29)$$

$$\omega_\sigma'' = -\frac{\sqrt{g}(1+4\sigma k_\sigma)}{4k_\sigma^{3/2}(1+\sigma k_\sigma)^{5/2}}, \quad (30)$$

The explicit expressions for the pure gravity waves on a clear free surface can readily be recovered by the present results as the thickness of ice-plate tends to zero. For the pure gravity waves ($h = \gamma = \sigma = 0$), $\partial^2\omega(k,0,0)/\partial k^2 < 0$ holds for all $k > 0$ and Eq. (28) simply reduces to

$$\zeta \simeq \frac{Mgt^2}{2^{5/2}\pi R^3}\exp\left(\frac{gt^2 z_0}{4R^2}\right)\cos\left(\frac{gt^2}{4R}\right). \quad (31)$$

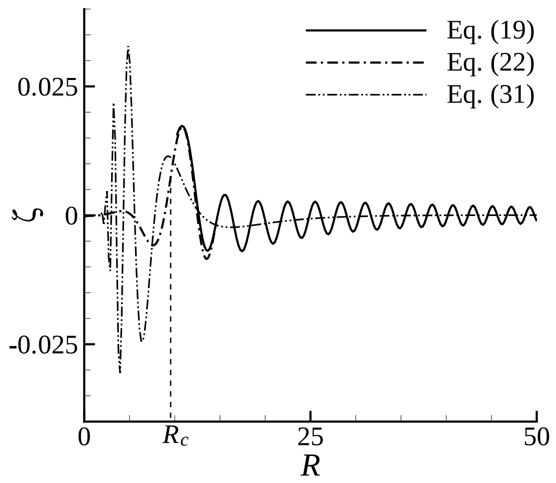


FIG. 2: Wave profile $\zeta$ with $M = 1$m, $h = 0.01$m and $z_0 = -0.5$m at $t = 5$s ($R_c = C_{g\min}t$).

It is found that there exists a minimal group velocity and the wave system observed depends on the moving speed of the observer. For an observer moving with the speed larger than the minimal group velocity ($R/t > C_{g\min}$), there exist two trains of waves, namely the long gravity waves ($k_1$) and the short flexural waves ($k_2$), the latter riding on the former. Moreover, the deflexions of the ice-plate for an observer moving with a speed near the minimal group velocity are expressed in terms of the Airy functions. An observer moving at a certain speed much lower than $C_{g\min}$ will see no wave motion at the observation point. Figure 2 shows the deflexions of the ice-water interface for different observers moving with a speed larger than, close to, the minimal group velocity, $C_{g\min}(0.01) \approx 1.91007$m/s.

As $\gamma = 0$, $\partial^2\omega/\partial k^2 < 0$ and Eq. (17) has one real positive root only, $k_\sigma$ for $\sigma \neq 0$ or $k_0$ for $\sigma = 0$. There is no minimal group velocity for the system and only one wave could be observed. It is noted that the submergence decay factor $\exp(gt^2 z_0/4R^2)$ for pure gravity waves will tend to zero since the wave-number $k_0$ tends to infinity as the field point approaches the source point ($R \to 0$). Therefore, we may find, on the basis of the analytical solution (31), a calm region on the free-surface that is near the submerged source point. This calm region will be enlarged due to the presence of an elastic plate, as shown in Fig. 2.

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