Passive trapping structures in the water-wave problem

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1. Introduction

Trapped modes are fluid oscillations of finite energy that are essentially confined to the vicinity of a structure. In the water-wave problem two types of trapped mode have been identified. Sloshing modes are fluid oscillations in the presence of a fixed structure, while motion modes are coupled to the motion of a freely-floating structure. The present work is concerned with modes that exert no net hydrodynamic force on the trapping structure and hence may be supported by both fixed and freely-floating structures. In the following, the conditions required for the existence of such "passive modes" are discussed and then the hydrodynamic properties of passive trapping structures are investigated.

2. Conditions for the existence of passive modes

Within the context of the linearized inviscid water-wave problem, it is known that certain special structures are able to support free oscillations with finite energy of the surrounding fluid - such oscillations are called trapped modes. The mathematical significance of such structures is that, at the frequency of the trapped mode, the solutions to certain forcing problems in the frequency domain may not be unique, or indeed may not exist, and this leads to numerical issues in the computation of the hydrodynamic coefficients for a trapping structure [1]. For a structure in open water, two types of trapped mode have been discovered and these are known as sloshing modes and motion modes.

A sloshing trapped mode is supported by a fixed structure so that a homogeneous Neumann condition is satisfied on its surface S. Thus, the velocity potential ϕ for the trapped mode satisfies

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S, \tag{1}$$

where n is a normal coordinate directed out of the fluid. The first sloshing trapping structures were discovered by M. McIver [2] and many more examples are now known. For the class of structures found in [2], it has been shown that the solution to a scattering problem exists, but that the solution to a radiation problem with the same symmetries as the trapped mode does not exist at the trapped-mode frequency. One consequence of the latter observation is that the corresponding added-mass coefficient is singular at the trapped-mode frequency.

In general, a sloshing trapped mode cannot be supported by a freely-floating structure that is able to respond to the hydrodynamic forces acting upon it. To fix ideas consider a structure of mass M that is constrained to move in heave with a velocity v so that the boundary condition on the structure is

$$\frac{\partial \phi}{\partial n} = v n_z \quad \text{on} \quad S, \tag{2}$$

where n_z is the z component of the normal to S and z is directed vertically upwards. The equation of motion for the structure is

$$\left[C - \omega^2 M\right] v = \omega^2 \rho \int_S \phi \, n_z \mathrm{d}S, \tag{3}$$

where, ω is the radian frequency of the oscillations, ρ is the fluid density, and the coefficient C includes the hydrostatic spring and the effects of any moorings (for the trapped mode to persist there must be no damping in the moorings). The right-hand side of equation (3) is proportional to the hydrodynamic force on the structure due to the trapped mode. For a moored structure, the possibility of finite-energy motions that satisfy (2) and (3) with non-zero ϕ and v has been known for some time and there have been a number of recent developments [4, 5]. The existence of motion trapped modes for structures without moorings has also been established [6, 7]. For motion trapped modes the hydrodynamic coefficients are all well behaved at the trapped-mode frequency.

In addition to the motion-trapped modes discussed above for which both the potential ϕ and the velocity v are non zero, there is the possibility that finite-energy solutions to (2) and (3) exist with

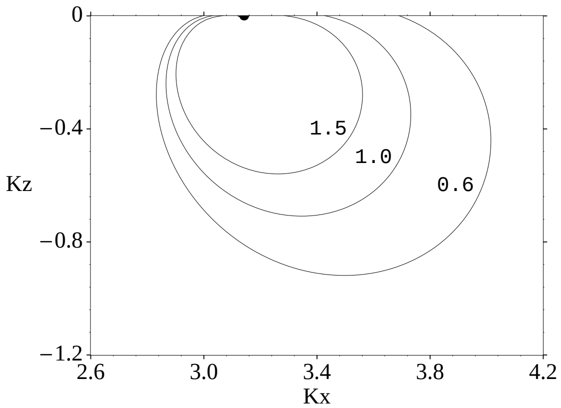

Figure 1: The right-hand element of passive trapping structures obtained from the equation $\psi(x,z;1) = \delta$ for $\delta \in \{0.6, 1.0, 1.5\}$.

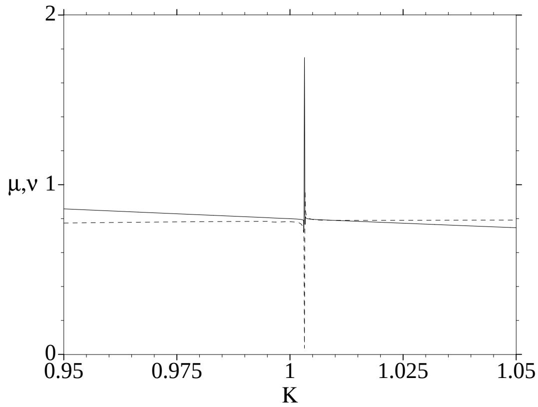

Figure 2: Non-dimensionalised added mass μ (—) and damping ν (−−) as a function of frequency K for the structure $\delta = 1.5$ with $n_p = 402$.

$\phi \neq 0$ but with $v = 0$ provided

$$\int_S \phi\, n_z \mathrm{d}S = 0. \tag{4}$$

A solution of this type corresponds to a fluid motion that satisfies the homogeneous condition (1) for a sloshing trapped mode, but for which the structure is able to float freely as the net hydrodynamic force on the structure due to the fluid motion is zero – structures that support modes of this type will be called here "passive trapping structures". Insight into how such structures might be found can be gained from examining the conditions under which (4) may be satisfied. A trapped mode does not radiate waves to infinity so that, for the particular case of two-dimensional motions in infinite depth,

$$\phi \sim \frac{\mu \cos\theta}{r} \quad \text{as} \quad r \to \infty \tag{5}$$

where r and θ are polar coordinates and θ is measured from the downward vertical. An application of Green's theorem to the trapped-mode potential ϕ and $u = z + g/\omega_0^2$, where ω_0 is the trapped-mode frequency, gives

$$\int_S \phi n_z \mathrm{d}S = \pi\mu \tag{6}$$

(note that this result requires $\partial\phi/\partial n = 0$ on S). Thus, equation (4) is satisfied if and only if $\mu = 0$, that is the coefficient of the dipole in the far-field expansion of ϕ is zero. When the velocity $v = 0$ the equations for ϕ are the same as those for a sloshing trapped mode, and it so happens that such modes have been constructed with $\mu = 0$ by Motygin & Kuznetsov [8].

3. Passive trapping structures

A family of two-dimensional passive trapping structures is shown in figure 1 and it corresponds to the potential

$$\phi(x,z) = \frac{1}{2K}\left(G_x(x,z;\pi/K,0) - G_x(x,z;-\pi/K,0)\right), \tag{7}$$

where $G(x,z;\xi,\zeta)$ is an infinite-depth source potential with the source located at (ξ,ζ), and $K = \omega_0^2/g$ denotes the infinite depth wavenumber corresponding to a trapped-mode frequency ω_0. Thus, $\phi(x,z)$ is the sum of two dipole potentials with the dipoles located at $\xi = \pm\pi/K$ on the free-surface. Since the dipoles have equal strengths and opposite directions the resultant wave-field will, at the trapped-mode frequency, be wave free with a vanishing far-field dipole coefficient. In the inverse procedure, utilised by [2] among others, the streamfunction ψ associated with ϕ is used to construct the trapping structures for a particular frequency ω_0 from the equation $\psi(x,z;K) = \delta$. The same approach can be adopted for a finite depth fluid domain; the only modifications necessary are in the form of the source potential G (and hence in the expressions for ϕ and ψ) and in the introduction of the finite depth

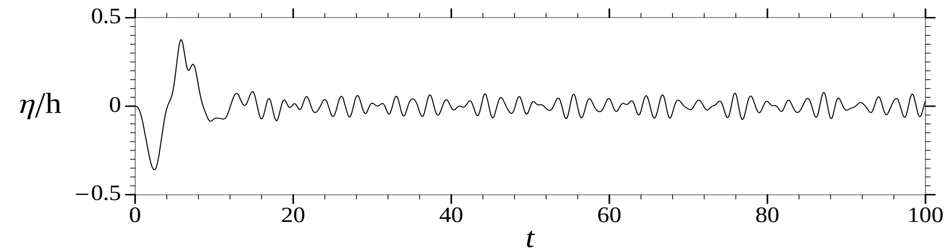

Figure 3: Free-surface elevation η/h at the mid-point of the internal free-surface as a function of time resulting from the prescribed displacement of the trapping structure.

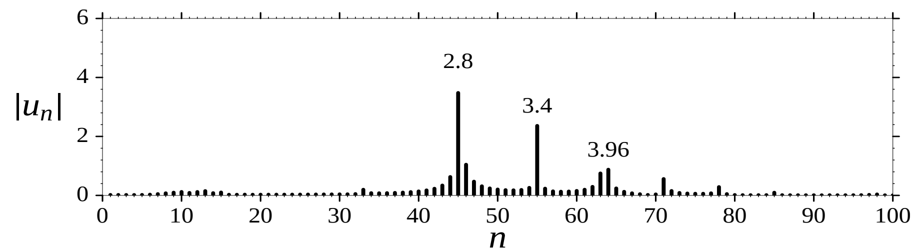

Figure 4: Discrete Fourier transform of the free-surface elevation shown in figure 3 with peak frequencies indicated. See [9] for further details on the transform.

wavenumber k which satisfies $k\xi = \pi$ and is related to K through $K = k \tanh kh$. The geometries of the resulting structures are very similar to those shown in figure 1.

Passive trapping structures support both sloshing and motion trapping modes because equations (1), (2) and (3) are all satisfied simultaneously by the (non-trivial) potential ϕ. The hydrodynamic properties of passive trapping structures are the same as for other motion trapping structures — both the radiation and scattering potentials exist at the trapped-mode frequency, despite the fact that structures also support sloshing trapped modes. Thus, the added mass and damping coefficient do not exhibit singular behaviour near that frequency. Seeking a numerical verification of these properties, a standard BEM frequency-domain code was used to compute the frequency dependence of the non-dimensionalised added mass μ and damping ν near the trapped mode wave number $K = 1$. The structure used in these calculations corresponds to the streamline $\delta = 1.5$ shown in figure 2. The inexact nature of the numerical approximation of the trapping structure is manifest in the very localised variations in μ and ν near the trapped-mode frequency; these are present because the discretisation of the trapping structure is actually a near-trapping structure with the corresponding radiation potential possessing a complex resonance located very close to the real-ω axis. As the number of panels n_p is increased the peaks reduce in width, increase in height, and converge on $K = 1$.

4. Time domain excitations

Excitation of sloshing trapped modes occurs when the trapping structure is forced to move in a prescribed fashion [9]. On the other hand, excitation of a motion trapped mode is possible if the motion trapping structure is given an inital velocity or displacement [6]. In both cases, the excitation corresponds to a singularity in the frequency-domain potential $\phi_R(\omega)v(\omega)$, either in the radiation potential ϕ_R itself or else in the velocity v. However, for a passive trapped mode the radiation potential is well-behaved at the trapped-mode frequency ω_0 and $v(\omega_0) = 0$ also, hence excitation of this mode is not possible using either excitation method mentioned above. To illustrate these excitation difficulties, a linear time-domain solver for two-dimensional motion in fluid of constant finite depth h was used to simulate the fluid and structure motion beginning from various initial conditons. For the computations here the particular trapping structure was chosen to have a trapped-mode frequency of $\omega\sqrt{g/h} = \sqrt{4\tanh 4}$; the shape of the structure is very similar to those shown in figure 1. The variation of the free-surface elevation at the mid-point of the internal free-surface is shown in figures 3 and 5 for two different simulations. Figure 3 shows the response to the forced displacement of the structure as prescribed by the function $Z(t)/h = (t/T)^3 e^{-\frac{t}{T}}$ whereas figure 5 shows the free-surface response

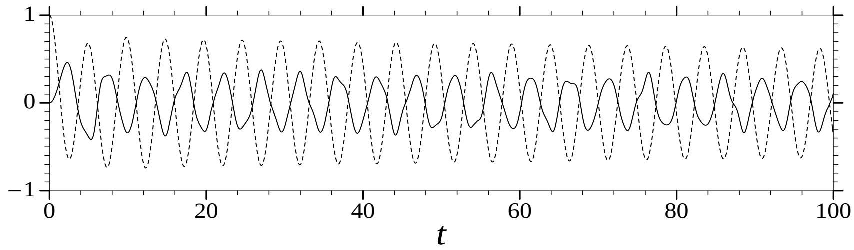

Figure 5: Variation of the free-surface elevation η/h (—) and displacement ζ/h of the structure (−−) at the mid-point of the internal free-surface with time after the release of the structure from rest.

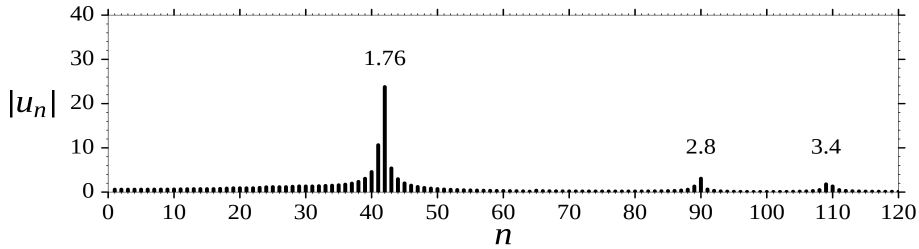

Figure 6: Discrete Fourier transform of the free-surface elevation from figure 5 with peak frequencies indicated.

for the coupled motion problem where the structure is given an inital displacement $Z(0)/h = 1$ and allowed to freely return to rest. In both cases, the discrete Fourier transform of the time signal shows that the trapped mode at the non-dimensional frequency $\omega\sqrt{h/g} \simeq 1.99933$ has not been excited. However, other weakly damped resonant modes are excited and these modes, although it is not easily discernible in the figures, decay slowly as time progresses.

Further investigations into the properties of passive trapping structures and modes are necessary. Thus far, only one structure has been investigated in the frequency and time-domains; a broader range of structures must be studied. A number of important questions regarding passive trapping structures have also yet to be answered: how can passive trapped modes be excited in the coupled water-wave problem?; how will the transient and long-time wave motions related to the trapped mode be affected by perturbations of the trapping structure? Addressing and answering these questions will be of foremost importance in future work on passive trapped modes.

5. References

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