Deterministic reconstruction and prediction of non-linear wave systems

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The aim of this study is to develop an efficient deterministic prediction model for irregular wave fields based on the exploitation of wave elevation time series, given by one or more probes. We use the High-Order-Spectral model (HOS) to simulate the wave field evolution numerically, in order to take the non-linear effects up to a desired order into account.

In this paper, we report on the development of an effective reconstruction scheme, for two dimensional wave fields and using one wave record, that allows us to get proper initial conditions for numerical simulations and nonlinear wave fields forecast.

1. Introduction

The phase averaged wave spectrum models, such as the well known Wavewatch model, have been the only practical prediction tools available since now. Despite their progress and success, the development of these so-called third generation models is limited by the inherent homogeneous and stationary assumptions, the linear transportation equation and the approximate source formulations for all physical processes. They are not able to predict deterministically the wave field evolution, and only statistical data representative of the sea state are available. But with the recent advances in computing technologies, the phase-resolved simulations of non-linear wave dynamics are becoming a feasible alternative to the spectral models for short term prediction. The objective of our work is therefore to develop a new deterministic prediction method, based on the use of the High Order Spectral (HOS) method, which we have been developing in the past few years, both for applications related to wave tank problems, and for non-linear simulations of sea states in unbounded domains (see e.g. Ducrozet *et al* [2]). Our approach for the deterministic prediction is very similar to the one reported in Yue *et al* [10] and Wu [12], from which this work is partially inspired.

A theoretical study was first conducted to provide the validity of the reconstructed wave field and its future predictability. We defined a reconstruction and prediction domain and we examined the effect various physical factors on the region. Easy ways to improve the zone were found.

One of the main key issues in developing our new prediction model was then to accurately reconstruct the initial wave field from the free surface elevation measured at one or several probes. High order simulation could only then been applied to predict the ocean wave field with optimal accuracy. For this, we used a multi-level iterative wave reconstruction process based on the free wave components of one probe record, with analytic solutions for first and second order reconstruction, and High-Order-Spectral non-linear wave model for upper orders. An optimization process was then applied to adjust the free wave components until the reconstruction error was minimized.

Finally, the wave reconstruction was tested with synthetic data obtained by a fully non-linear HOS simulation. Good agreement was found for second order 2D case reconstruction, for different wave steepness. However, for high steepness, it showed that high-order effects inclusion in wave reconstruction is of significance.

2. Predictable domain

Reconstruction and prediction can only be considered in a specific time-space zone that depends on the wave conditions and the number of probes used [12]. For a linear ocean wave field with no ambient current and fixed probes, we show that the predictable region only depends on the measurement period *T*, the slowest and fastest group velocities of the wave components crossing the probe, and the spreading angle range for a multidirectional wave field. When considering real waves, as the nonlinearities modify the group velocity, the predictable domain changes, according to the interactions between all the wave components in the field. Nonlinear interactions turn out to increase the domain.

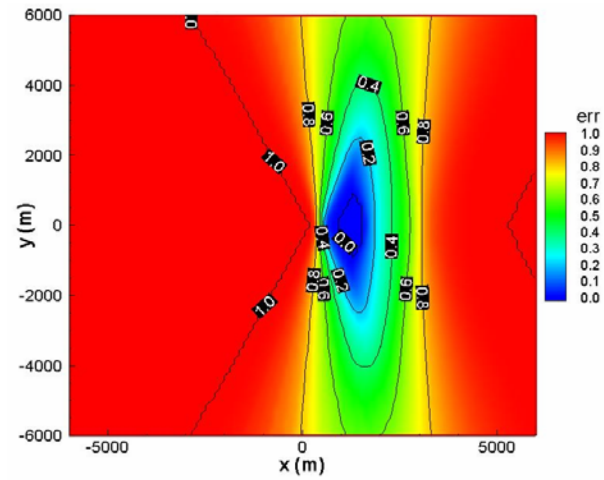

Figure 1 : Predictable zone and error at t >T for a 3D wave field, based on a measurement at the origin, during a period of T=300s

The effect of probe motion, ambient current and finite water depth on the predictable domain have also been examined respectively for a unidirectional field. We found that probes moving with constant speed against the waves can significantly increase the domain, which suggests a method to improve predictability for ship-mounted sensors. A following current also improves the predictability, whereas an opposite current decreases it. We noticed that the effect of finite water depth on the predictable zone is not monotonous and depends on the frequency band of the wave spectrum.

Finally, we studied the effect of combining probe records on the domain. We have shown that the combined zone is much larger that the sum of individual predictable regions based on the same probe. This suggests another way to improve predictability easily, as multiple probes are often used when measuring multidirectional seas.

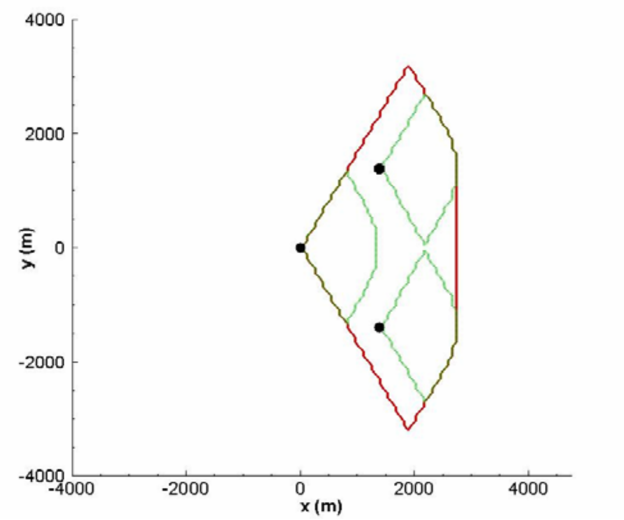

Figure 2 : Association of three probes located at the black points; individual (green) and combined (red) predictable regions at t=T for a 3D wave field

3. High Order Spectral method

For medium wave steepness, the non-linear interactions play an important role in the wave dynamics and cannot be ignored in the wave field reconstruction and prediction. Rather than employing an analytic solution, which is very complex and costly to obtain, we choose to apply a very powerful numerical method for calculating the wave field, the HOS (High Order Spectral) method.

The HOS model is a pseudo-spectral method that follows Zakharov's equations. It takes N modes into account and their non-linear interactions up to an arbitrary order M in wave steepness. The method has spectral convergence with respect to N and M and the use of Fast Fourier Transforms reduces the computational cost to MNlog(N) at each time step.

The governing equations for the method are Laplace's equation for the velocity potential and the fully non-linear free surface boundary conditions written in surface quantities, the single-valued surface elevation $\eta(x,t)$ and the surface potential $\phi^s(x,t) = \phi(x, z = \eta, t)$.

$$\Delta\phi = 0, z \le \eta(x,t)$$

$$\frac{\partial \phi^s}{\partial t} = -g\eta - \frac{1}{2}\left|\nabla\phi^s\right|^2 + \frac{1}{2}\left(1+\left|\nabla\eta\right|^2\right)\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta}$$

$$\frac{\partial \eta}{\partial t} = \nabla\phi^s.\nabla\eta + \frac{1}{2}\left(1+\left|\nabla\eta\right|^2\right)\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta}$$

The two unknowns are expressed at collocation points of a periodic domain and further time-marched once the vertical velocity has been obtained through the solution of a Dirichlet problem for the potential. The latter is solved by the HOS expansion of the potential in orders of the wave elevation in parallel with the order consistent formulation of West *et al* [11].

Since the solution of the stated non-linear boundary problem is totally determined by the initial conditions for the surface elevation and the surface potential, one key issue of wave reconstruction and forecasting is though to determine $\phi^s(x, t=0)$ and $\eta(x, t=0)$ that satisfy $\eta(x_{probe}, t) = \tilde{\eta}(x_{probe}, t)$ for $t \in [0, T]$, where $\tilde{\eta}(x_{probe}, t)$ is the probe record and T the measurement period.

4. Reconstruction of a unidirectional wave field based on a single probe record

4.1. Outlines

The first step in wave reconstruction is to determine the frequency-direction spectrum of the wave field from the records. For the two-dimensional case, this can be easily done using a Fourier transform. The prediction region can then be evaluated and the reconstruction process may begin.

As suggested by Yue *et al* [10] and Wu [12], we use an iterative optimization process for reconstruction. It starts with a linear analytic model and increases the non-linear order gradually. This allows us to study wave fields with various steepness without employing large computational efforts when not needed: for small wave steepness, the procedure may stop at a low order avoiding costly simulations associated with high-orders of nonlinearity.

The reconstruction is based on the optimization of the record free wave components. For the second order reconstruction, we use a conventional analytic mode coupling method to calculate the wave-wave interactions (see Dalzell [1]) and the iterative process of Duncan and Drake [3] to decouple non-linear effects from the free wave components.

For orders up to two, the initial wave surface elevation and surface potential are calculated from the free wave components obtained at the previous order. The wave field is then reconstructed and its non-linear evolution is simulated with the HOS method during the measurement period, so that we obtain the reconstructed wave records. An optimization procedure is finally applied to adjust the free wave components until the reconstruction error is minimized in the reconstruction domain.

If the accuracy is good enough, the process ends, otherwise the order of nonlinearity is increased by one and the new model is applied to the reconstructed wave field.

After the wave field is reconstructed, it can be forecasted by using either an analytic wave solution calculated from the free wave components directly, or a HOS simulation with the reconstructed field at $t=T$ as initial condition. Note that only the wave field within the predictable region can be deterministically forecasted.

4.2. Reconstruction error

The wave field reconstruction error is defined by the following formula, where N_p is the number of probes used (N_p = 1 in our case) and $W(x_{probe},t)$ a weighting function that allows us better to capture the features at wave peaks in wave reconstruction.

$$\varepsilon = \left\{ \frac{1}{N_p} \sum_{1}^{N_p} \int_0^T W(x_{probe},t) \left[\eta(x_{probe},t) - \tilde{\eta}(x_{probe},t) \right]^2 dt \right\}^{1/2}$$

Thus the reconstruction becomes an optimization process to minimize ε and have the best accuracy for the reconstructed wave field.

4.3. Second order reconstruction

The surface elevation at probe position is assumed to comprise the sum of first and second order components:

$$\eta^{probe}(t) = \eta^{(1)}(t) + \eta^{(2)}(t)$$

This decomposition cannot be completed directly as the free wave components are required to evaluate the second order interactions, but are accurately obtained if they previously have been decoupled from the non-linear effects. We thus apply Duncan and Drake's cascade procedure to get the free wave components, where the second order elevation at step $q+1$ is calculated from the decoupled first order components at step q by a mode coupling method. The decomposition stops when convergence is obtained.

$$\eta^{(1)}_{q+1} = \tilde{\eta}^{probe}(t) - \eta^{(2)}_{q+1}$$
$$\eta^{(2)}_{q+1} = \eta^{(2)}(\eta^{(1)}_{q})$$

Special attention had to be paid at high frequencies (>2rad/s) where a numerical instability problem occurs for medium and high steepness. As the measured amplitudes get very small at high frequencies, an unrealistic build up of first and second order components takes place if the procedure is used without any constraint, which makes the calculations diverge.

As suggested by Duncan and Drake, we tried to control this instability by setting a cut-off frequency ω_c above which the first order amplitudes are set to zero and thus the second order amplitudes are set to the measured amplitudes. Duncan and Drake's solution consists in cutting-off at the frequency where the second order amplitude modulus exceeds the measured amplitude modulus, rather than choosing an arbitrary ω_c.

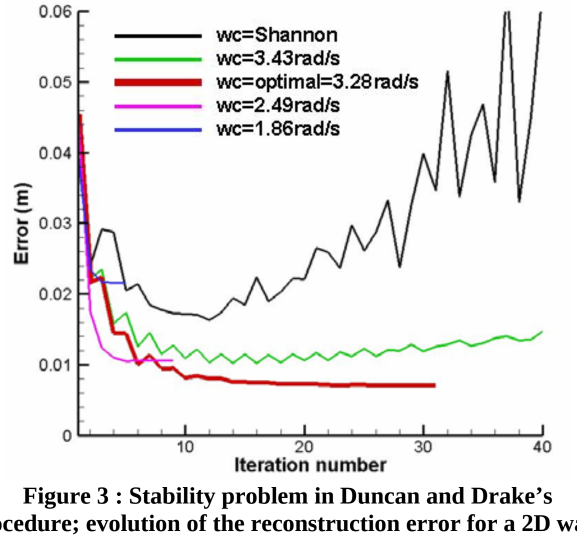

Figure 3 : Stability problem in Duncan and Drake's procedure; evolution of the reconstruction error for a 2D wave field during the process for various cut-off frequencies

Because of the roughness of our amplitudes spectra (due to the frequency discretization), we couldn't manage to determine ω_c this way. Moreover, we found that a very small change in ω_c could have a significant influence on the quality of the reconstructed record (see Figure 3). Therefore, rather than trying every suitable frequency as ω_c, which would have been very costly, we chose to use an optimization process to determine the cut-off frequency that minimizes the reconstruction error at probe position.

4.4. Validation of second order reconstruction at t=T/2

The wave reconstruction model was tested with synthetic data obtained by fully non-linear HOS simulation. We chose to use Tanaka's procedure [9] to create a wave field from a 2D-JONSWAP spectrum, and simulated its evolution with the HOS code up to a desired order of nonlinearity.

Different wave steepness were studied. On Figure 4, we put in relation the reconstruction error ε evaluated at probe position to the error on the reconstructed surface elevation and potential on the entire reconstruction zone. The latter was evaluated as the discrepancy to the synthetic data at each point of the discretized space domain and averaged.

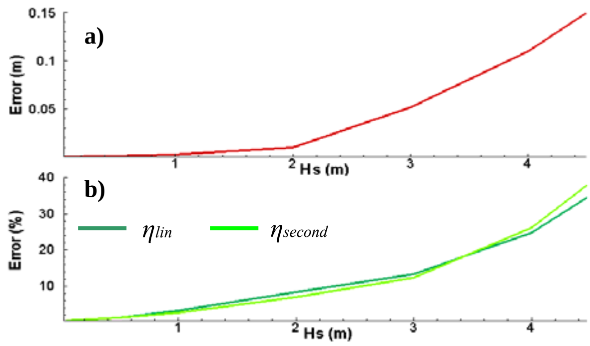

Figure 4 : 1st and 2nd order reconstructions for 2D wave fields with various steepness, a) Reconstruction error ε after Duncan & Drake's decomposition, b) Error at t= T/2 between theoretical and reconstructed surface elevations in the reconstruction zone

Good agreement was found for wave fields with steepness lower than 3%. We note a significative improvement of the peaks reconstruction compared to the linear solution. For upper steepness, high-order effects become too important and the second order solution doesn't take all nonlinearities into account any more; the linear solution seams even better than the second order one.

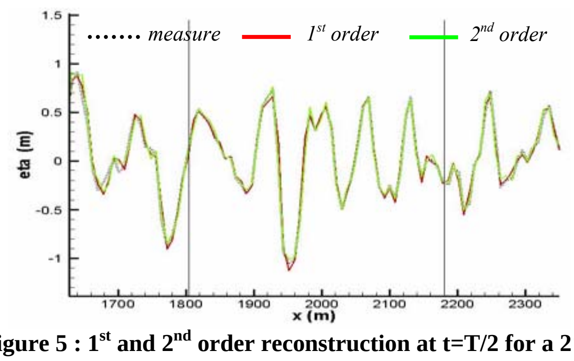

Figure 5 : 1st and 2nd order reconstruction at t=T/2 for a 2% steepness wave field, from one probe measurement at x=2000m; black lines delimit the reconstruction zone

Note that we also reconstructed the surface potential which is needed as initial condition by the HOS code for second order wave prediction and upper orders reconstruction.

4.5. Optimization of free wave components for high order probe record reconstruction

As we do not have any explicit form for the wave solution, we have to simulate the wave field from 0 to T with the HOS method to get the probe record each time the reconstruction error is evaluated. This is very costly, so we concentrated on very efficient optimization methods.

We first tried direct search schemes, also called derivative free methods because they don't use the derivatives of the objective by the optimization variables to progress. Therefore, no explicit expression for the error in terms of the free wave components is required.

The first procedure tested was the Simplex method of Nelder and Mead [4] which showed to be very slow. We then explored the set of directional approaches such as Rosenbrock's [7], Swann's [8] or Powell's [6] methods. Swann's and Powell's optimization processes turned out to be useless as we couldn't bracket the minimum before the maximum displacement authorized was reached for the optimization variables. As a matter of fact, it's necessary to limit the possible values for the free surface elevation components if we want the spectra to remain physical. For the Rosenbrock's method, we used Palmer's improvement for direction's updating in order to avoid colinearity [5].

Figure 6 shows the convergence of both Simplex and Rosenbrock's methods for a third order reconstruction scheme. The latter appears to be the most efficient.

We also considered derivatives-based optimization methods, such as the Conjugate Gradient method and the Quasi-Newton method. Unfortunately those methods induce tremendous computational efforts as they require as many objective evaluations as optimization variables used (which means as many HOS simulations) for any gradient evaluation (with a decentred finite difference method). Other methods for the gradient calculation exist, like the adjoint method, but they are very complex and have not been implemented yet. Therefore we restricted our study to the procedures presented before, based on the function evaluation rather than its gradient.

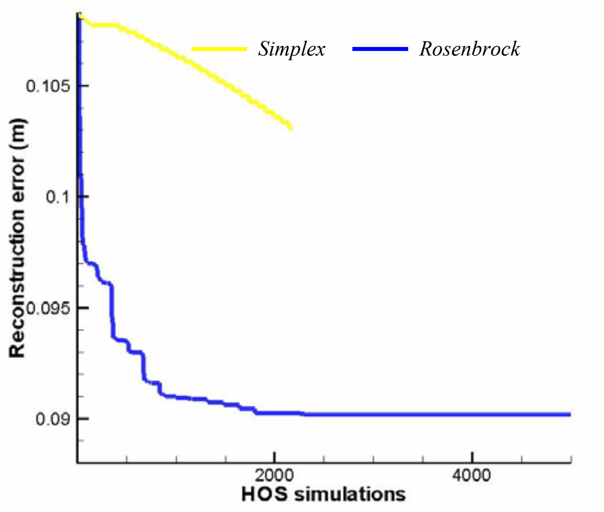

Figure 6 : Simplex and Rosenbrock's optimization results for the third order reconstruction of a 0.5% 2D wave field

Results presented in Figure 6 mainly aim at selecting the most appropriate optimization algorithm for our application. The optimization procedure based on HOS simulations is still under development. Especially, one may notice on the latter figure that the reconstruction error obtained with Rosenbrock's method is still very important and much bigger than the one obtained with second order Duncan and Drake's algorithm. We are presently working on this.

5. Conclusion

In this paper, we have presented significant results on the development of an efficient reconstruction method for irregular non-linear wave-fields using one probe wave record of limited duration. Second order reconstruction has been validated with several synthetic data and further experimental validations in our wave basin are planned. Regarding higher orders reconstruction, we are still improving our optimization procedures.

At least, the scheme is also being extended to directional seas, and will take benefit from the recent parallelization of the code.

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