Trapping structures with linear mooring forces

by J. N. Newman
<jnn@mit.edu>

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1 Introduction

In the linear theory of wave interactions with floating bodies, trapping structures demonstrate examples of nonuniqueness. Most work on this topic is focused on fixed bodies, and on the existence of nontrivial solutions of the homogeneous boundary-value problem for the velocity potential. McIver [1] established the existence of fixed trapping structures in two dimensions. Her approach has been used subsequently to show that a variety of these structures exist in both two and three dimensions. McIver and McIver [2,3] have shown that similar structures exist which support trapped modes when they are freely floating and moving in an oscillatory manner. To distinguish between these two complementary problems, the terms 'sloshing trapping structure' and 'motion trapping structure' are introduced in [2] and [3].

The two-dimensional constructions in [1] and [2] are based on tracing the streamlines generated by pairs of symmetrical singularities on the free surface which are separated by one-half wavelength (or, more generally, an integer plus one-half wavelengths). Thus there are no radiated waves in the far field. For the sloshing trapping structure a pair of wave sources is used. For the motion trapping structure, wave sources and wave-free singularities are combined, such that the total dipole moment vanishes in the far field, and it follows from Green's theorem that non-zero body motions can exist without an exciting force.

In the present work a connection is made between these two complementary problems by considering the more general case where a floating body is restrained by a linear restoring force, represented by a nondimensional restoring coefficient (or 'spring constant') k. In the limit $k = 0$, the body is free and the family of motion trapping structures is recovered. Conversely, if $k \to \pm\infty$, the body is fixed and sloshing trapping structures are recovered. Structures with different profiles are found in the intermediate regimes $k < 0$ and $k > 0$.

The construction for the more general case uses a linear combination of the singularities required in the two complementary limits. One minor change from the analysis in [2] is the use of quadrupoles to represent the component associated with motion trapping structures, instead of the combination of sources and wave-free singularities. That particular combination, with zero dipole moment in the far field, is equivalent to a vertical quadrupole defined as the second z-derivative of the source. Thus we consider here the streamlines and body profiles generated by pairs of sources and quadrupoles. If the source strength is zero, motion trapping structures are generated; if the body is fixed, sloshing trapping structures are generated. In the intermediate regime where a finite restoring force exists, different trapping structures are found. To simplify the analysis we restrict our attention to vertical (heave) motion of a two-dimensional body in a fluid of infinite depth.

2 Analysis

The motion is harmonic with frequency ω in the plane x, z, where $z = 0$ is the free surface and z is positive upwards. The coordinates are nondimensionalized with respect to the wavenumber $K = \omega^2/g$. Thus the wavelength is 2π and the free-surface condition is $\phi - \phi_z = 0$ on $z = 0$. We define the complex variable $Z = z + \mathrm{i}x$ and the complex potential $F = \phi + \mathrm{i}\psi$, where ϕ is the velocity potential and ψ the stream function.

For a pair of point sources of oscillatory strength, situated at $x = \pm\pi/2$ on the free surface, the complex potential can be expressed in the form

$$F_s = \mathrm{e}^{(Z+\mathrm{i}\pi/2)}\mathrm{E}_1(Z + \mathrm{i}\pi/2) + \mathrm{e}^{(Z-\mathrm{i}\pi/2)}\mathrm{E}_1(Z - \mathrm{i}\pi/2) - 2\pi\mathrm{H}(\pi/2 - |x|)\mathrm{e}^Z. \tag{1}$$

Here E_1 is the exponential integral, defined as in [4], and $\mathrm{H}(\pi/2-|x|)$ is equal to 1 if $|x| < \pi/2$, otherwise zero. The harmonic time-dependence has been factored out, taking advantage of the fact that the usual out-of-phase components are canceled by the half-wavelength spacing. For a pair of quadrupoles at the same points, the complex potential is

$$F_q = \frac{\partial^2 F_s}{\partial Z^2}. \tag{2}$$

As $|Z| \to \infty$, $F_s \simeq 2Z^{-1}$ and $F_q \simeq 4Z^{-3}$. Thus F_s is 'dipole-like' in the far field, and F_q is a higher-order singularity with zero dipole moment. Note that from the physical viewpoint (1) represents a pair of sinks with negative flux, since $F_s \simeq -\log(Z \mp \mathrm{i}\pi/2)$ as $Z \to \pm\mathrm{i}\pi/2$.

Following the analysis of a freely-floating structure in [2], the solution of the equation of motion for heave will admit nontrivial homogeneous solutions if (a) there is no damping, and (b) the sum of the inertial and restoring forces is zero. Zero damping is ensured for any motions associated with the singularities (1) and (2), since there is no radiated wave energy. The second condition corresponds to the equation (cf. [2], equation 4)

$$\rho g W(1 + k) - \omega^2(M + a) = 0. \tag{3}$$

Here $\rho g W$ is the hydrostatic restoring force with ρ the fluid density and W the width of the waterplane, and k is a nondimensional external restoring coefficient. M is the body mass and a the added mass. From Green's theorem it can be shown ([2], equation 18) that

$$\rho g W - \omega^2(M + a) = -\pi\mu\rho\omega^2, \tag{4}$$

where μ is the far-field dipole moment, defined such that the asymptotic approximation of the potential ϕ_0 for unit heave velocity is

$$\phi_0 \simeq -\mu K \mathrm{Re}(Z^{-1}) \quad \text{as} \quad |Z| \to \infty. \tag{5}$$

Thus a necessary condition for a trapping mode to exist is

$$k = \pi\mu K/W. \tag{6}$$

3 Streamlines and body profiles

We consider the streamlines ψ =constant, associated with the complex potential

$$F = SF_s + QF_q - V(Z + 1). \tag{7}$$

Here S is the source strength, Q the quadrupole strength, and V is the heave velocity. The parameters S, Q, V are real, and nondimensionalized by ω and g. From (5) and (6) it follows that

$$k = -(2\pi S/KWV). \tag{8}$$

Without loss of generality it can be assumed that $V \geq 0$. Thus, for a positive restoring coefficient, the source strength S must be negative (corresponding to a source with positive flux). Conversely, if $S > 0$ (a sink), the mooring restraint is inertial.

Streamlines which surround the singular points $x = \pm\pi/2$ below the free surface correspond to the profiles of trapping structures moving with vertical velocity $(g/\omega)V$. By tracing the streamlines numerically, it can be shown that such profiles do in fact exist, for most but not all combinations of the parameters S, Q, V.

If all three parameters are nonzero, Q and V must have the same sign. The variety of possible structures is illustrated in Figure 1, for $Q = V$ and $|S| + Q = 1$. In all cases there is at least one streamline which surrounds the singular point $x = \pi/2$ in the domain below the free surface, defining the profile of a trapping structure. For small values of S and k this profile is similar to the case $S = 0$. In this regime one profile exists, with a stagnation point where it intersects the dividing streamline. As $|S|$ increases the stagnation point and dividing streamline shift toward the right ($S > 0$) or left ($S < 0$), with more significant changes. For $S = 0.7$ (approximately) the right side of the body profile is vertical at the free surface and for $S > 0.7$ the vertical component of the interior normal vector is positive at all points. In this regime trapping structures exist which satisfy one of John's uniqueness requirements for fixed structures, that no vertical lines starting in the free surface intersect the body. On the other hand, for $S \leq -.7$ the stagnation point is on the vertical axis and the body profile which intersects this point is continuous with its reflection in $x < 0$; thus a single closed body is represented without an interior free surface, satisfying the other uniqueness requirement of John. In both cases, for larger values of $|S|$, a family of additional streamlines surround the singular point within the interior of the profile which intersects the stagnation point; in the limit $S \to \pm 1$ this family corresponds to the profiles of the fixed trapping structures in [1].

If $V = 0$ the body is fixed. For $Q = 0$ there is a family of profiles, as in [1]. Similar profiles exist for $Q/S > 0$. There is apparently no upper bound on Q/S, but the domain $Q/S > 10$ has not been explored. Similar profiles exist also for small negative values of Q/S, but this range is limited to, approximately, $-0.6 < Q/S < 0$. Within these ranges, sources and quadrupoles can be combined to generate fixed trapping structures which generalize the results in [1]. The principal effect of the quadrupole is to induce a stagnation point on the innermost body profile.

If $V > 0$ the body is moving. For $S = 0$ there is a single profile for each value of $Q/V > 0$, as shown in [2]; in that case the body is free, with no external restraint. It is not necessary to include quadrupoles if the body is moving and restrained. For $Q = 0$ and small positive values of S/V there is one profile with a stagnation point for each value of S/V; for large positive values of S/V there is a family of profiles similar to the $V = 0$ case. For small negative values of S/V there is no profile, but for $S < -0.8V$ a profile exists with no interior free surface, similar to the lower left plot in Figure 1. Note that in the latter case the source flux is positive, and the streaming flow is downward. It is remarkable that a closed body is formed in this case, on the downstream side of the sources.

A corresponding analysis has been made for the axisymmetric three-dimensional case, extending the results in [3]. The streamlines and profiles are qualitatively similar to those shown in Figure 1. Computations of the damping and added-mass coefficients for these structures confirm that the equation of motion is homogeneous.

It is not surprising that a wide variety of moored trapping structures exist. There are many examples of bodies with zero heave damping at a particular frequency, and for the corresponding values of the added mass, displaced mass, and waterplane area a suitable value of the restoring coefficient k can be found such that (3) is satisfied. Analogous results have been derived by Kyozuka and Yoshida [5], for resonant motions of bodies generated by wave-free singularities.

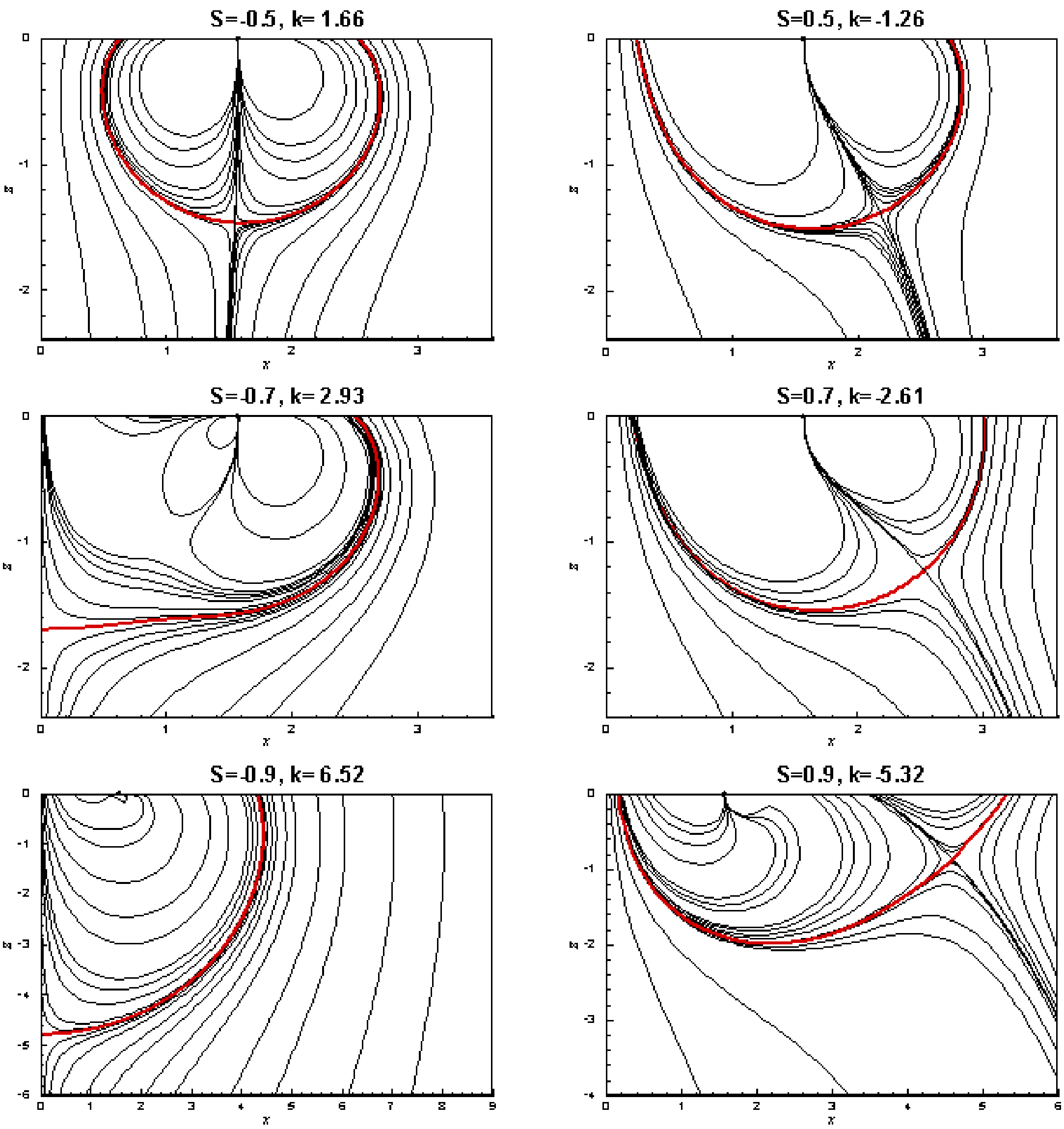

Figure 1: Streamlines generated by the complex potential (7) with $Q = V = 1 - |S|$. In the left column $S < 0$ and $k > 0$, and conversely in the right column. The body profile is shown by the heavy line. The dividing streamline intersects this profile normally at the stagnation point. The singular point $x = \pi/2$ is marked by a filled circle. Reflected streamlines for $x < 0$ are not shown.

References

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