Examples of motion trapped modes in two and three dimensions

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1. Introduction

The term “motion trapped mode” (MTM) was introduced by Phil McIver at the 20th Worskhop to distinguish it from the conventional trapped mode describing the local oscillation of a fluid around a *fixed* body in an unbounded fluid. An example of an MTM is the motion which can occur in the Venice gates as described by C.C. Mei at the 19th Workshop. He showed that a persitent free motion of N buoyant bottom-hinged gates could occur in the absence of a incident wave field. In McIver & McIver (2006) two conditions are provided which must be satisfied for an MTM to exist at a given frequency in either two or three dimensions. First the wave damping should vanish at that frequency and secondly, the inertia forces, involving the body’s inertia (including its added inertia) should balance any spring restoring force such as the hydrostatic force at the same frequency. Phil went on to use an inverse method devised originally by Maureen McIver (1996) in producing examples of fixed body trapped modes, to construct a pair of ‘mirror image’ surface-piercing cylindrical sections (cylinders, for short) in two dimensions which exhibited the phenomenon of an MTM in heave motions at a particular frequency. The idea was also extended to produce an example of a partly immersed axisymmetric MTM.

If such a body, or pair of cylinders in two dimensions, were displaced initially from equilibrium and then released, they would ultimately settle down into a continuing harmonic oscillation at that frequency. Of course in practice viscous damping, neglected in the theory, would ensure a slow decay in time.

In this paper we present three examples of bodies exhibiting MTMs, two in two dimensions including, as far as we are aware, for the first time an example involving submerged cylinders, and one in three dimensions being an axisymmetric heaving motion. In contrast to the inverse methods used to date, each of the examples involve simple geometrical shapes and the search for MTMs was prompted by heuristic arguments.

2. A heuristic approach to finding motion trapped modes

The search for MTMs in is based on the following simple approximate argument. Consider the scattering of an incident wave train of frequency $\omega/2\pi$ by a *single* two-dimensional cylinder which is free to respond to the incident wave in a single mode of motion, opposed only by a linear restoring force. Suppose that for a particular wave frequency and parameter values of the cylinder, it is found that the wave is totally reflected so that the reflection coefficient R_1 satisfies $|R_1| = 1$. We now imagine an identical freely floating cylinder a (suitably large) distance $2b$ upstream of the first, moving as its mirror image in the line midway between them. The reflected wave will in turn be totally reflected and it is easily shown that a symmetric standing wave or MTM can be set up if the spacing satisfies $R_1 = \mathrm{e}^{2ikb}$ where k is the wavenumber, or

$$kb = -\tfrac{1}{2}\arg\{R_1\} + n\pi \qquad (1)$$

n an integer. If the two cylinders are assumed to move exactly in antiphase when heaving or exactly in phase when rolling or swaying the condition for an antisymmetric MTM between them requires an extra $\pi/2$ on the right hand side of (1).

This wide-spacing argument which neglects the influence of the local field near one cylinder on the other, provides an approximate formula for MTMs as well as providing a starting point for the location of MTMs when local effects are not ignored. The wide-spacing argument has been exploited successful in many examples of *fixed* body trapped modes, many of which have been reported at this workshop.

Thus the task of finding MTMs for pairs of bodies has been replaced by the search for bodies which reflect all incident wave energy at some particular frequency when allowed to move in response to the waves.

3. Scattering by a cylinder in free response to incident waves

The theory for determining the reflection and transmission of incident waves (say R_1 and T_1) by bodies allowed to respond in either heave, sway or roll, and whose motion was opposed by a linear restoring force was presented in Evans & Linton (1989), who were investigating the use of a submerged buoyant tethered cylinder for use as an 'active wave reflector'. They obtained, via the use of various reciprocal relations, the compact expressions (equations (24) and (25) of their paper, noting a $\mathrm{e}^{\mathrm{i}\omega t}$ time variation)

$$R_1 = \frac{(CR \pm \mathrm{i}T)}{(C-\mathrm{i})}, \qquad T_1 = \frac{(CT \pm \mathrm{i}R)}{(C-\mathrm{i})} \quad (2)$$

where the upper(lower) sign refers to heave(sway or roll) motions. Here R, T are the reflection and transmission coefficients for the cylinder held *fixed* in the same incident wave,

$$C = ((M+I)\omega^2 - \lambda)/B\omega \quad (3)$$

where M, B are the added mass and damping coefficients for the cylinder in its mode of motion, I is the mass (or moment of inertia) of the cylinder, and $\omega/2\pi$ is the incident wave frequency. Finally λ is the constant of proportionality between any external force (or moment) on the cylinder and its displacement (or angular displacement). Because of the assumed symmetry it can be shown that $R/T = \mathrm{i}\chi$ where χ is real and we see that the transmission coefficient T_1 will vanish if $C = \pm\chi$ or

$$\lambda = ((M+I)\omega^2 \mp B\omega\chi \quad (4)$$

whence $R_1 = R \mp T$ and $|R_1| = 1$.

3.1 Submerged tethered cylinders

It was shown in Evans & Linton that equation (4), with the lower sign, can indeed be satisfied for a submerged tethered buoyant circular cylinder opposed in its motion by the horizontal component of the tension in its tethers. In this case $\lambda = M'(1-s)g/ls$ where M' is the mass of water displaced by the cylinder, s is the specific gravity of the cylinder and l is the length of the cable securing the cylinder to the bottom. It follows that equation (1) provides an approximate expression for both symmetric and antisymmetric MTMs between two identical submerged tethered buoyant swaying circular cylinders.

3.2 Floating cylinders

In the case of floating cylinders in two-dimensional motion, the (hydrostatic) restoring force is given by $\lambda = W\rho g$ where W is the waterplane area occupied by the cylinder. For a semi-immersed circular cylinder allowed to move in heave response to incident waves (in deep water) numerical tests suggest that T_1 is never zero. The theory outlined above was also adapted to include the possibility of wave reflection by a cylinder in combined heave *and* sway motions and in this case also, it was determined numerically that no zeros of transmission could occur. See figure

1 to see curves of transmission coefficients. One serious deficiency of this particular test is that there is only one free non-dimensional parameter, ka, to vary (a being the cylinder radius).

In contrast, cylinders of rectangular cross-section offer an extra degree of freedom in the choice of rectangle aspect ratio. A numerical investigation revealed that T_1 could vanish either for wide or narrow cylinders. In fact there is an independent heuristic argument which can be used to motivate the case for the existence of frequencies at which total reflection occurs in the case of narrow rectangular cylinders in heave. Over 30 years ago, Evans & Morris (1972) showed that two fixed closely-spaced parallel vertical surface-piercing barriers could reflect all wave energy at particular frequencies. Subsequently, Newman (1974) used a method of matched asymptotic expansions to confirm these results. In the narrow gap between the barriers the fluid motion was represented as a rigid-body slug-type flow. Thus Newman's analysis is also applicable to the heave motion of a narrow rectangular cylinder.

3.3 Axisymmetric cylinders

The final example of an MTM involves the free heave motion of a thick-walled floating circular shell. The argument here uses the fact that two-dimensional rectangular cylinders in heave response are capable of reflecting all incident waves. Then there is an axisymmetric version of the two-dimensional wide-spacing arguments used to derive (1) – see Shipway & Evans (2003), for example, in which b becomes the radial distance to the structure and an extra factor of $\pi/4$ is added to (1).

4. Exact theory and results

The conditions for MTMs have been given by McIver & McIver (2006) and are, in our notation,

$$(M^w + I)\omega^2 - \lambda = 0 \qquad \text{and} \qquad B^w = 0$$

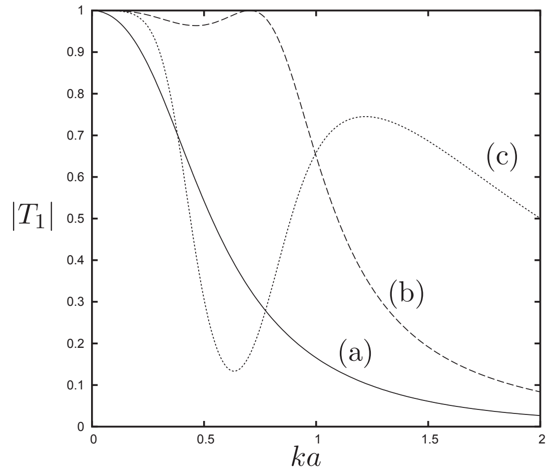

Figure 1: Transmission coefficient against ka for a half-immersed circular cylinder, radius a, in deep water: (a) fixed; (b) free to respond in heave only; (c) free to respond in heave and sway.

Here M^w and B^w are the added mass (or inertia) and damping coefficients for the double-body combinations in two dimensions or for the axisymmetric thick-walled circular cylinder. The superscript w is used to suggest that in the two dimensional examples the double-body can be replaced, from symmetry arguments, by a single body next to a vertical wall on which either a Neumann or Dirichlet condition is to be satisfied.

We only provide a brief overview of the various semi-analytical methods used to calculate the solutions to the various problems we have discussed. For problems involving submerged circular cylinders, finite depth multipole potentials are used. For circular cylinders in the free surface, infinite depth dipoles and wave-free potentials are used. For rectangular cylinders, eigenfunction expansions are used to derive integral equations whose solutions are approximated using variational methods. In each case, numerical results are obtained to a high degree of accuracy.

In fig. 1 the absence of zeros of transmission are illustrated for a semi-immersed cylinder of circular cross-section both constrained and free to move. In fig. 2, the exact parameters computed for MTMs for a pair of floating rectangular cylinders are shown. In this problem and its three-dimensional ax-

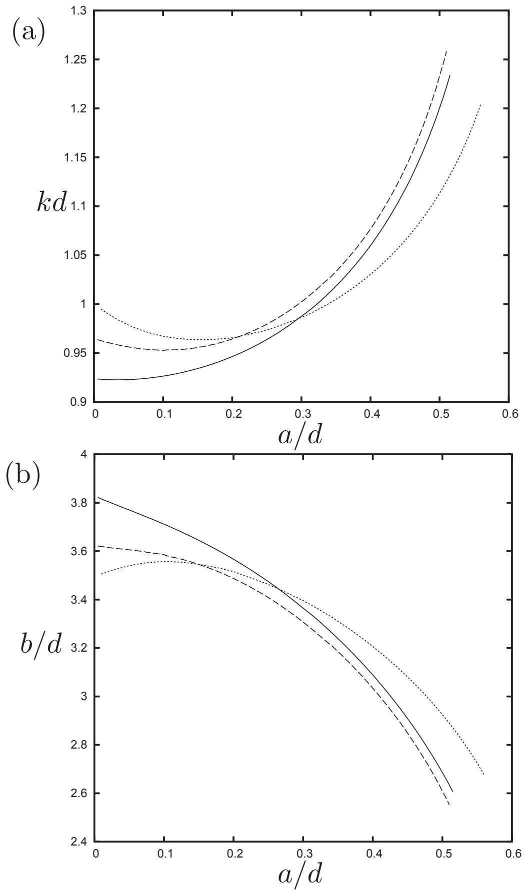

Figure 2: Variation of (a) kd and (b) b/d with a/d giving rise to MTMs for a pair of rectangular cylinders (width $2a$, draught d, spacing between cylinders $2b$) in heave. Draught to water depth ratios of $d/h = 0.1$ (solid), 0.2 (dashed), and 0.4 (dotted).

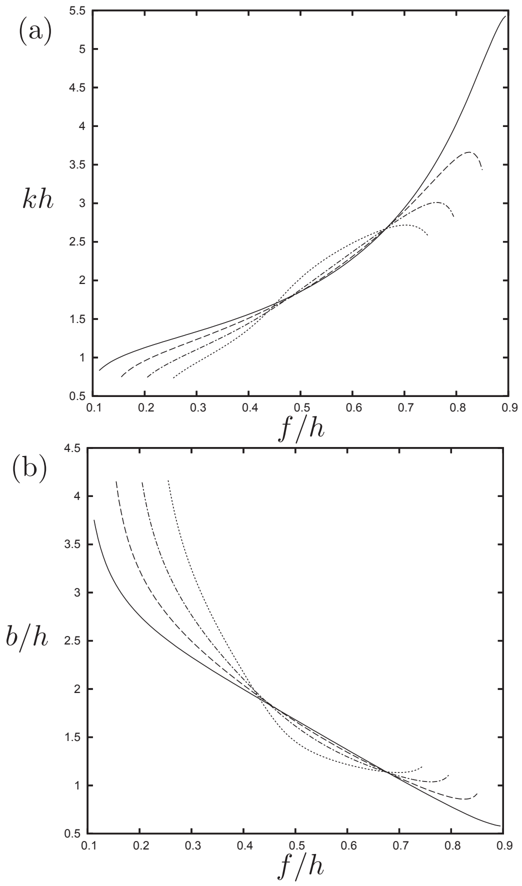

Figure 3: (a) Variation of (a) kh and (b) b/h with f/h for MTMs involving pairs of tethered submerged circular cylinder (water depth h, submergence $f = h - l$, specific gravity $s = 0.06$, spacing $2b$) Cylinder radius ratios of $a/h = 0.1$ (solid), 0.15 (dashed), 0.2 (chained), 0.25 (dots).

isymmetric counterpart, the wide-spacing approximation turns out to be a poor predictor of exact parameters for MTMs. In contrast, wide-spacing arguments give a very good approximation to the exact parameters for pairs of submerged tethered cylinders, shown in fig. 3. More detailed results will be presented at the Workshop.

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