A semi-analytical formulation for the wave-current interaction problem with a vertical bottom-seated cylinder including square velocity terms

by

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1 General

This paper treats the diffraction problem by a bottom-seated, surface-piercing uniform circular cylinder in a wave and current coexisting field. The main contribution of the present is the inclusion of the square of current's velocity U^2. Therefore the problem's solution is appropriate to be used for higher current velocities and can be extended for floating axisymmetric bodies moving in a wave field with relatively higher speed. Linear potential theory is applied and the solution method utilizes an appropriate semi-analytical formulation of the potential terms which need to satisfy the inhomogeneous form of the boundary condition on the free surface. The resulting Sturm-Liouville problems are treated properly using the one dimensional Green's function which is common for all potentials multiplied by nonzero powers of current's velocity. The main objective of the present work is to assess the contribution that U^2 has on the hydrodynamic forces acting on the cylinder for relatively higher Froude numbers $Fn = U/\sqrt{gb}$ and to investigate the change on the free surface elevation after including the relevant terms.

2 Mathematical formulation and solution

When dealing with wave-current problems or higher-order hydrodynamic problems using potential theory, the main challenge is the proper treatment of the free surface boundary condition. The existing efforts that approach the solution to the problem analytically apply the linear potential theory and they end at the linear current velocity term U (or $\tau = U\omega/g$ after normalization). Characteristic examples are the works reported by Zhao and Faltinsen (1989), Nossen et al. (1991), Emmerhoff and Sclavounos (1992) and Malenica et al. (1995). The interaction problem of structures with waves and current has been treated up to the second-order for the waves but still using the linear current term. The reader can refer to the work of Büchmann et al. (1998) who solved the problem numerically. Here the problem is extended by taking into account the square of the current's velocity which requires an additional term in the expansion of the total velocity potential of the wave field. This term is formulated as $\varphi = \varphi^I + \varphi_0^{(1)} + \mathrm{i}\tau\varphi_1^{(1)} + \tau^2\varphi_2^{(1)}$ where the various parts in the right hand side of the previous expression denote the incident wave, the zero-*th* order diffraction potential and the first- and the second-order perturbation potentials respectively. The (1) indicates that φ (which is the spatially dependent term) is the linear potential that depends on the frequency of encounter ω. The associated problem up to the calculation of $\mathrm{i}\tau\varphi_1^{(1)}$ is well treated in the literature. Nevertheless, the inclusion of $\tau^2\varphi_2^{(1)}$ in the formulation requires further elaboration of the free surface boundary condition. To this end, the total velocity potential in the combined waves-current field is defined as $\Phi = U\phi_0 + \phi$ where $\phi_0 = \overline{\phi} + r\cos(\theta - \alpha)$. $\overline{\phi}$ is the perturbation of the current by the body and α is the current's heading angle. The wave field is governed by the time dependent potential ϕ which is expanded in perturbation series according to $\phi = \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + O(\varepsilon^3)$, where $\phi^{(n)} = \mathrm{Re}\{\varphi^{(n)}(r,\theta,z)e^{-ni\omega t}\}$, $n = 1,2$ and

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$\varphi^{(1)}$ has been defined previously. For n=2, $\varphi^{(n)}$ expresses the second-order velocity potential due to the wave action that depends on the double of the frequency of encounter 2ω. Using a Taylor series expansion around the mean undisturbed free surface and retaining only the $O(H/2)$, $O(U)$, $O(UH/2)$, $O(U^2)$, $O((H/2)^2)$, $O(U^2H/2)$ terms, the original condition on the free surface is decomposed at $O(\varepsilon)$ and $O(\varepsilon^2)$ according to

$$-\nu\varphi^{(1)}+\varphi_z^{(1)}-2\mathrm{i}\tau\nabla\phi_0\nabla\varphi^{(1)}+\left(U^2/g\right)\nabla\phi_0\nabla\left(\nabla\phi_0\nabla\varphi^{(1)}\right)+\left(U^2/2g\right)\nabla\varphi^{(1)}\nabla\left(\nabla\phi_0\nabla\phi_0\right)=0 \quad (1)$$

$$\begin{aligned}-4\nu\varphi^{(2)}+\varphi_z^{(2)}&=\left(\mathrm{i}\omega/g\right)\left(\nabla\varphi^{(1)}\nabla\varphi^{(1)}-\frac{1}{2}\varphi^{(1)}\left(\varphi_{zz}^{(1)}-\nu\varphi_z^{(1)}\right)\right)-\left(\omega/g\right)\tau\nabla\phi_0\nabla\left(\varphi^{(1)}\varphi_z^{(1)}\right)\\&+4i\tau\nabla\phi_0\nabla\varphi^{(2)}-\left(1/4\omega\right)\tau\nabla\phi_0\nabla\left(\nabla\varphi^{(1)}\nabla\varphi^{(1)}\right)+\left(1/2\omega\right)\tau\nabla\varphi^{(1)}\nabla\left(\nabla\phi_0\nabla\varphi^{(1)}\right)\end{aligned} \quad (2)$$

It is noted that $H/2$ is the amplitude of the incoming waves. Eq. (1) is the first-order boundary condition on the free surface including nonlinear current terms while Eq. (2) is the same boundary condition at second-order with respect to the wave action which also incorporates the effect of the linear current. Here, only the first equation is considered while the second is given for future reference. The solution method employed in the present is based on the semi-analytical formulation for the first- and second-order perturbation potentials $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$. This method has been successfully used in the past for solving the body-wave interaction problems to first- and second-order (Garret, 1971; Huang and Eatock Taylor, 1996; Mavrakos and Chatjigeorgiou, 2006) and it is now extended to capture the body-wave interaction problem in the presence of a current field. This method results to Sturm-Liouville problems which require the construction of the associated one dimensional Green's function. For the uniform circular cylinder examined in the present, this function has the same form at all orders.

It is evident that the derivation of $\varphi_2^{(1)}$, which constitutes the main contribution of the present work, requires the knowledge of all successive components in the expansion $\varphi=\varphi^I+\varphi_0^{(1)}+\mathrm{i}\tau\varphi_1^{(1)}+\tau^2\varphi_2^{(1)}$ including the first-order perturbation component. The calculation of $\varphi_1^{(1)}$ does not creates major difficulties as the function that expresses the radial dependence of the associated inhomogeneous term in the free surface boundary condition is easy to be derived and it converges immediately during numerical implementation. On the contrary, the calculation of the second-order perturbation term is much more difficult and exhibits different asymptotic behaviour which is characterized by its slower convergence. The final relation that provides the second-order perturbation potential can be written as

$$\varphi_2^D=-\mathrm{i}\omega_0\frac{H}{2}h\sum_{m=-\infty}^{\infty}\mathrm{i}^m\sum_{j=0}^{\infty}Z_j(z)Z_j(h)\left[\int_1^{\infty}\frac{b^2}{h^2}\frac{\zeta}{b}Q_m^{(2A)}\left(\frac{\zeta}{b}\right)G_{mj}\left(\frac{r}{b};\frac{\zeta}{b}\right)\mathrm{d}\left(\frac{\zeta}{b}\right)+\int_1^{\infty}\frac{b^2}{h^2}\frac{\xi}{b}Q_m^{(2B)}\left(\frac{\xi}{b}\right)G_{mj}\left(\frac{r}{b};\frac{\xi}{b}\right)\mathrm{d}\left(\frac{\xi}{b}\right)\right]e^{\mathrm{i}m\theta} \quad (3)$$

where h is the water depth, b is the radius of the cylinder, ω_0 is the frequency of the undisturbed incident waves, G_{mj} is the one dimensional Green's function and Z_j are the orthogonal eigenfunctions in vertical direction. In the scope of the present contribution the most difficult part is to express analytically and to programming the radial functions $Q_m^{(2A)}$ and $Q_m^{(2B)}$. Although the radial function $Q_m^{(2B)}$ is significantly long and complicated, its calculation requires only the determination of the zero-*th* order perturbation component φ_0^D. Thus, at each radial distance r away from the body, the relevant integral is calculated only once. On the contrary, the integral that involves $Q_m^{(2A)}$ in Eq. (3), it virtually represents a two-stage integration to infinity. In addition, the derivative of Green's function is required introducing additional difficulties as more asymptotic values need to be determined.

3 Numerical results and discussion

The solution method that was described briefly above was applied for a bottom seated cylinder with radius equal to the water depth, $h=b$. Numerical results are presented for the normalized horizontal exciting forces $F_x/\rho g b^2 H/2$ as well as for the wave elevation $\eta/H/2$ in a sufficient distance away from the body. The exciting forces are given for two Froude numbers (0.1 and 0.2) while the wave elevation is depicted for Fn=-0.2 and k_0b=1.0 where k_0 is the wave number of the incident waves. For drawing conclusions with regard to the contributions of the linear and the nonlinear current terms, the curves that depict the variation of the hydrodynamic forces have plotted together. In addition, the part of the total hydrodynamic force due to $\tau^2\varphi_2^{(1)}$ component is given separately. The current velocities that correspond to the selected Froude numbers can be considered relatively high. It should be mentioned that for lower current velocities, for Fn=±0.05 for instance, the contribution of U^2 to the hydrodynamic loading is relatively insignificant. Worth-mentioning differences are observed from Fn=±0.1 and above (Figs. 1 and 3). As expected the nonlinear perturbation term contributes more for higher values of U. This is immediately apparent for Fn=±0.2. The contribution of the nonlinear current term to the total hydrodynamic force acting on the cylinder, rises from approximately 0.18 to 0.5 (max nondimensional values in Figs. 2 and 4) for Fn=±0.1 to Fn=±0.2 respectively. An important point which must be underlined is that for positive Froude numbers, i.e. when the current and the incoming waves have the same direction, the second-order perturbation component reduces the total hydrodynamic loading. A reverse tendency is observed when the current is opposite to the direction of the waves, i.e. for negative Froude numbers. It should be also noticed that the U^2 term of the total hydrodynamic force (which is depicted separately in Figs. 2 and 4) affects primarily its magnitude. By inspecting the associated curves in Figs. 1 and 3, it can be easily concluded that their trend remains unaffected.

Indicative examples for the free surface elevation around the body are showing in Figs. 5 and 6. These correspond to k_0b=1.0 with a Froude number that equals to 0.2. The direction of waves is from negative to positive Y while the current is opposite. Fig. 5 shows the pattern of the wave elevation taking into account only the $O(\tau)$ term of the potential while in Fig. 6 both $O(\tau)$ and $O(\tau^2)$ terms were considered. From the first sight the contours seem to be identical as no evident differences are observed. The only easily detected difference concerns the maximum value of the wave elevation which is approximately equal to 1.8 of the wave amplitude when the complete problem is considered (Fig. 6) and slightly lower when the nonlinear perturbation term is removed (Fig. 5). Nevertheless the location of the maximum elevation for both cases is on the same area and on the lee side of the cylinder with respect to the direction of the waves. This is due to the fact that the opposite direction of the current amplifies the wave elevation. A different behaviour is observed when the waves and the current hit the cylinder from the same angle and Fn = 0.2. In this case the maximum elevation occurs very close to body, while the inclusion of the nonlinear current term reduces the height of the diffracted waves on lee side.

4 References

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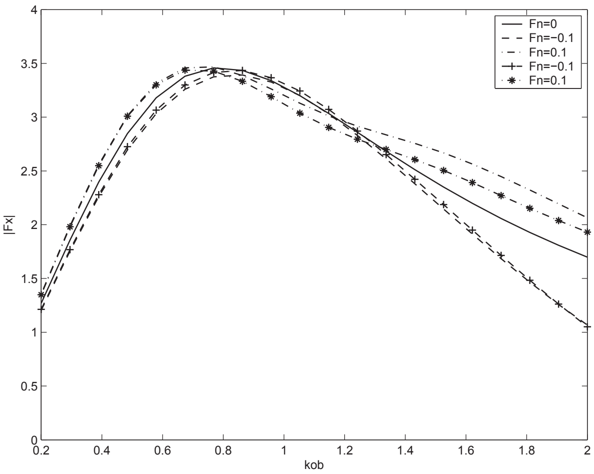

Fig. 1. Horizontal exciting force on the cylinder for *Fn*=0.1 and *Fn*=-0.1. Symbols are used to denote the contribution of both *U* and U^2.

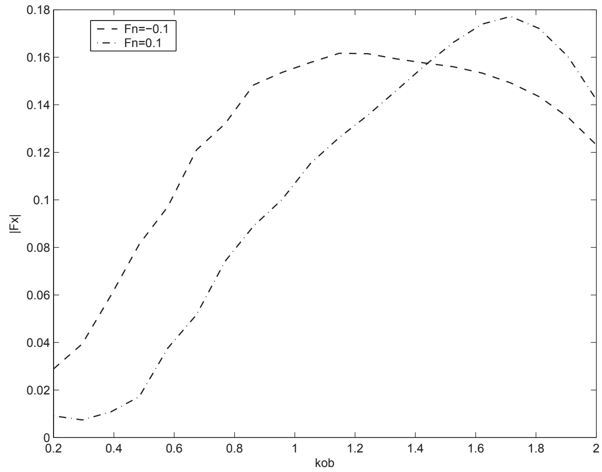

Fig. 2. Contribution of the second-order perturbation component on the horizontal exciting force on the cylinder for *Fn*=0.1 and *Fn*=-0.1.

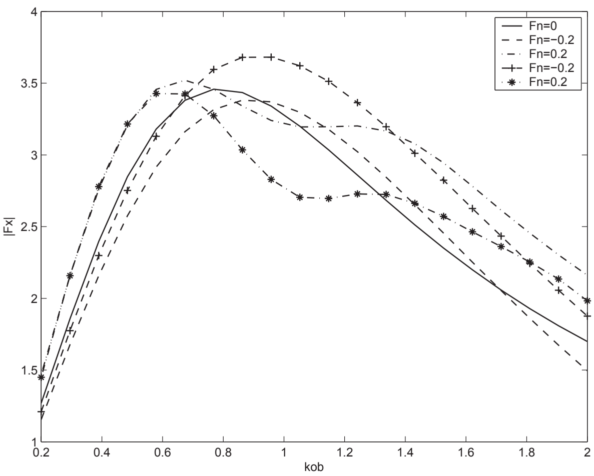

Fig. 3. Horizontal exciting force on the cylinder for *Fn*=0.2 and *Fn*=-0.2. Symbols are used to denote the contribution of both *U* and U^2.

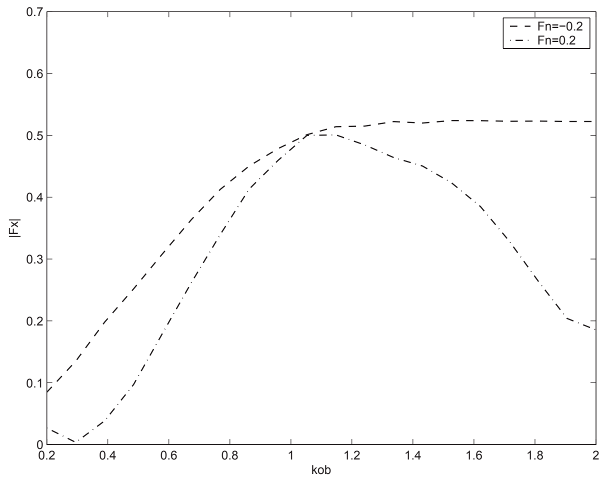

Fig. 4. Contribution of the second-order perturbation component on the horizontal exciting force on the cylinder for *Fn*=0.2 and *Fn*=-0.2.

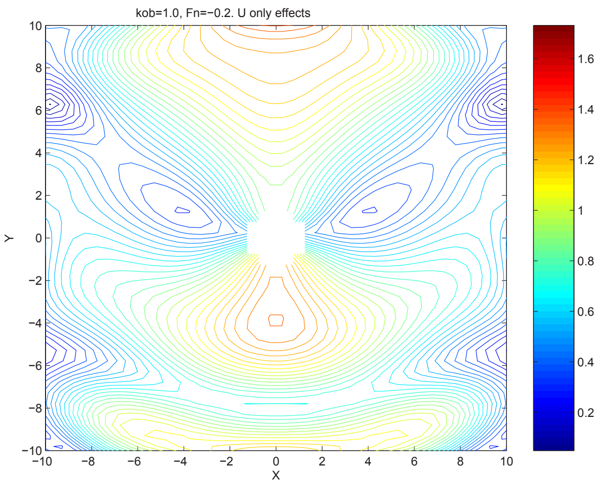

Fig. 5. Free surface elevation using only linear current component.

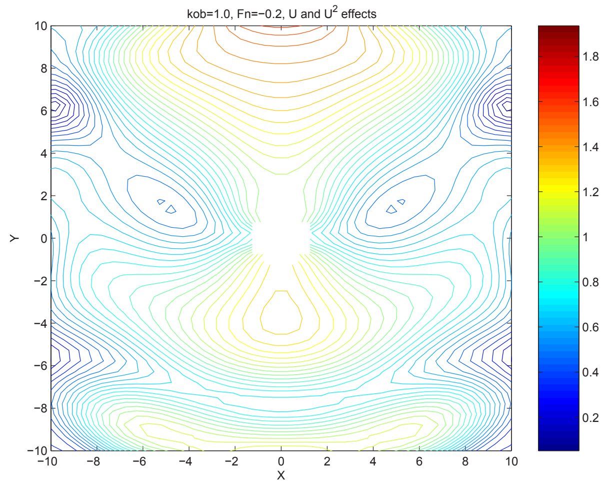

Fig. 6. Free surface elevation using both linear and nonlinear current components.