Hamilton's principle for dissipative systems and Wagner's problem

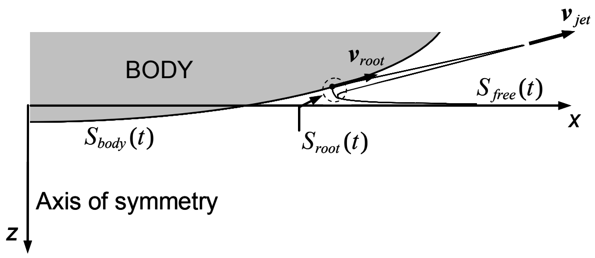
LEONARDO CASETTA
lecasetta@ig.com.br

CELSO P. PESCE
ceppesce@usp.br

Fluid-Structure Interaction and Offshore Mechanics Laboratory
Mechanical Engineering Department
Escola Politécnica
University of São Paulo
05508-900, São Paulo, Brazil

1 Introduction

As well known, within Wagner's approach to the water impact problem, the usual Dirichlet's condition at the free surface can be asymptotically interpreted as the infinite frequency limit of the linear floating body problem (see, e.g. Mei, 1983, p. 11; Newman, 1977, p. 298; Faltinsen, 1990, p. 286). This motivates us to extend to the water impact problem the Hamilton's principle postulated by Miloh p. 231, 1984, there in the context of the oscillating floating body at a free surface (FBP). For simplicity, we restrict our analysis to the vertical impact case (VWIP). To the authors' knowledge, the novelty is related to the proper consideration of the dissipated energy through the jet root in the Hamiltonian, as done with the wave damping in the FBP, what leads to a corresponding Rayleigh's dissipation function. This is achieved with the aid of a possible interpretation of the added mass as an *explicit* function of the body penetration depth, as in Pesce, 2003, p.754[1].



2 The field equation and nonlinear boundary conditions

Analogously to the Lagrangian postulated by Miloh, 1984, p. 231, we define

$$L = L^* - \Im = -\rho \iiint_{V(t)} (\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + g\eta)\mathrm{d}V + \tfrac{1}{2}\rho \iint_{S_{root}(t)} \phi(\boldsymbol{v}_{jet} - \boldsymbol{v}_{root})\cdot\boldsymbol{n}\mathrm{d}S - \Im, \qquad (1)$$

as an extended form appropriate to treat the nonlinear VWIP. The last term, $\Im$, can be interpreted as the dissipation function given in terms of the flux of energy through the jet root, i.e.

[1] Alternatively, the added mass may be viewed as an explicit function of time; see Miloh, 1991, p. 45.

$$\frac{\mathrm{d}\Im}{\mathrm{d}t} = \tfrac{1}{2}\rho \iint_{S_{root}(t)} \phi_t (\boldsymbol{v}_{jet} - \boldsymbol{v}_{root}) \cdot \boldsymbol{n}\mathrm{d}S^{\,2}. \tag{2}$$

The application of Hamilton's principle, $\int_{t_1}^{t_2}(\delta L + \delta W)\mathrm{d}t = 0$, in Eq. (1), where $\delta W = F\delta z$ is the virtual work done by the vertical impact force F and z is the body penetration depth, does recover the nonlinear problem, given by

$$\varphi\phi = 0 \text{ , in } V(t), \tag{3}$$

$$\phi_t + \tfrac{1}{2}\nabla\phi \cdot \nabla\phi + g\eta = 0, \text{ in } S_{free}(t), \tag{4}$$

$$\phi_n - v_n = 0, \text{ in } S_{body}(t) \text{ and } S_{free}(t), \tag{5}$$

$$F = -\iint_{S_{body}(t)} p\mathrm{d}S, \text{ in } S_{body}(t). \tag{6}$$

This is true if Eq. (2) is assumed to be valid *a priori*, enabling the vertical impact force to be obtained from a classical procedure within analytical mechanics.

3 The vertical impact force within Wagner's theory

Within Wagner's theory, both, the contact area of the body and the free surface elevation are considered to be collapsed onto the horizontal plane (see, e.g., Scolan and Korobkin, 2001, p. 294), and so Eq. (4) becomes $\phi_t = 0$ at the free surface, $S_{free}(t)$, excluding the jet root, $S_{root}(t)$, what implies $\phi = 0$ at $S_{free}(t)$ (see, e.g., Pesce, 2005, p. 396, for a discussion on this issue). Thus, the linearized form of the Lagrangian (see, equivalently, Miloh, 1984, p. 232) is given as

$$L_L^* = \tfrac{1}{2}\rho \iint_{S_{body}(t)\cup S_{free}(t)} \phi\phi_n \mathrm{d}S = \tfrac{1}{2}\rho \iint_{S_{body}(t)} \phi\phi_n \mathrm{d}S = T_{bulk}, \tag{7}$$

where $T_{bulk} = \tfrac{1}{2} M_{bulk}\dot{z}^2$ is the kinetic energy associated to the bulk of the fluid (excluding the jets), being M_{bulk} the corresponding added mass (see Casetta and Pesce, 2006, for a deeper discussion on this subject).

From the Lagrange equation for linearly dissipative systems and with $\Re = \mathrm{d}\Im / \mathrm{d}t$, the vertical impact force expression can be written as

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{bulk}}{\partial \dot{z}} + \frac{\partial T_{bulk}}{\partial z} - \frac{\partial \Re}{\partial \dot{z}}. \tag{8}$$

A possible and interesting way to overcome an apparent difficulty in the derivation of $\Re$ with respect to $\dot{z}$ is to apply a form of Hamilton's principle given in McIver, 1973, for open systems.

[2] The operator $\mathrm{d}/\mathrm{d}t$ ought to be interpreted with respect to the relative velocity between the material surface (jet) and the control surface (jet root) (see Irschik and Holl, 2004, for a comprehensive discussion on the application of Reynold's transport theorem to movable control surfaces).

Applying Hamilton's principle for open continuous systems

As a flux of energy exists from the bulk of the fluid to the jet, an open system approach may be followed, as in McIver, 1973. The virtual work principle applied to the Wagner's problem then reads

$$\delta T_{bulk} + F\delta z - \rho \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} (\nabla\phi \cdot \delta\boldsymbol{r})\mathrm{d}V + \rho \iint_{S_{root}(t)} (\nabla\phi \cdot \delta\boldsymbol{r})(\boldsymbol{v}_{jet} - \boldsymbol{v}_{root}) \cdot \boldsymbol{n}\mathrm{d}S = 0 . \tag{9}$$

The corresponding Hamilton's principle for the VWIP can be directly obtained by integration of Eq. (9) with respect to time, i.e.

$$\int_{t_1}^{t_2} \delta T_{bulk}\mathrm{d}t + \int_{t_1}^{t_2} F\delta z\mathrm{d}t + \rho\int_{t_1}^{t_2} \iint_{S_{root}(t)} (\nabla\phi \cdot \delta\boldsymbol{r})(\boldsymbol{v}_{jet} - \boldsymbol{v}_{root}) \cdot \boldsymbol{n}\mathrm{d}S\,\mathrm{d}t = 0 . \tag{10}$$

The energy balance expression then follows from Eq. (10) by letting the virtual displacements coincide with actual displacements, i.e. $\delta\boldsymbol{r} = \nabla\phi\mathrm{d}t$, and by using Reynold's transport theorem in the sense of Irschik and Holl, 2002, what leads to

$$\frac{\mathrm{d}T_{bulk}}{\mathrm{d}t} + F\dot{z} + \rho \iint_{S_{root}(t)} \tfrac{1}{2}\nabla\phi \cdot \nabla\phi(\boldsymbol{v}_{jet} - \boldsymbol{v}_{root}) \cdot \boldsymbol{n}\mathrm{d}S = 0 . \tag{11}$$

Recalling that $F = -\frac{\mathrm{d}}{\mathrm{d}t}(M_{bulk}\dot{z})$ and, as $M_{bulk} = M_{bulk}(z(t))$, then $\dot{z}^2 \frac{\partial}{\partial z}(M_{bulk}) = \dot{z}\frac{\mathrm{d}}{\mathrm{d}t}(M_{bulk})$ (see Pesce, 2003, p. 754), it may be concluded from Eq. (11) that the time rate of kinetic energy transfer through the jet root does not depend on the shape of the body and is given by[3]

$$\rho \iint_{S_{root}(t)} \tfrac{1}{2}\nabla\phi \cdot \nabla\phi(\boldsymbol{v}_{jet} - \boldsymbol{v}_{root}) \cdot \boldsymbol{n}\mathrm{d}S = \tfrac{1}{2}\frac{\mathrm{d}M_{bulk}}{\mathrm{d}t}\dot{z}^2 . \tag{12}$$

From Eq. (12) and recalling the boundary condition at the jet root, i.e. $\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla\phi = 0$, we have a proper choice for $\Re = \mathrm{d}\Im / \mathrm{d}t$, satisfying Eq. (2), as

$$\Re = \frac{1}{4}\frac{\mathrm{d}M_{bulk}}{\mathrm{d}t}\dot{z}^2 . \tag{13}$$

From the point of view of Hamilton's principle for variable mass particles

By using $F = -\frac{\mathrm{d}}{\mathrm{d}t}(M_{bulk}\dot{z})$ and $T_{bulk} = \frac{1}{2}M_{bulk}\dot{z}^2$, Eq. (10) transforms into

$$\int_{t_1}^{t_2} \delta T_{bulk}\mathrm{d}t + \int_{t_1}^{t_2} F\delta z\mathrm{d}t + \rho\int_{t_1}^{t_2} \tfrac{1}{2}\frac{\mathrm{d}M_{bulk}}{\mathrm{d}t}\dot{z}\delta z\mathrm{d}t = 0 . \tag{14}$$

Equation (14) is a particular form of the general Hamilton's principle proposed by Mušicki, 2000, p. 1067, when applied to the simpler case of mass dependent only on position. In the special case of constant impact velocity an 'added mass potential' may be defined within Mušicki formalism as $\wp = f(z) = -\frac{1}{2}M_{bulk}(z)u_0^{\,2}$, such that the Lagrangian function may be rewritten as

[3] See the particular case of elliptic contact lines in Scolan and Korobkin, 2003.

$L' = L_L^* - \wp = T_{bulk} - (-\frac{1}{2} M_{bulk} u_0{}^2) = 2T_{bulk}$. This leads to Hamilton's principle in the form $\int_{t_1}^{t_2} \left(2\delta T_{bulk} + \delta W\right) dt = 0$. In other words, in the case of constant impact velocity, half the kinetic energy transferred to the liquid goes to the jet, as often emphasized by several authors (see, e.g., Cointe et al., 2004). Moreover, the system is holonomic.

It may be also shown (to be submitted) that the relation between the flux of kinetic energy and the explicit variation of the added mass with position, equivalently given by Eqs. (12), (10) or (14), is in fact a particular case of a more general result within particle dynamics, derivable from the concept of 'fictitious particles' (see Irschik and Holl, 2002, *apud* Truesdell and Toupin, 1960).

4 Conclusions

Within Wagner's theory, it was shown that when the added mass is assumed to be explicitly dependent on the penetration depth, Hamilton's principle may be consistently applied by taking into account the dissipated energy through the jet root. The vertical impact force can be recovered from Hamilton's principle by interpreting the problem as a dissipative one and relating a Rayleigh's dissipation function with the flux of energy through the jet root.

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