# Scattering of water waves by arrays of arbitrary bodies


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## 1 Introduction

Scattering of water waves by large arrays of floating or submerged bodies is of considerable importance in off-shore engineering as well as climate research. While the standard method of solution of the full three-dimensional linear diffraction problem requires numerical methods involving the discretisation of the surfaces of all bodies, interaction theories may be used if the bodies are sufficiently widely spaced. However, even then the resulting linear system of equations is of considerable dimension if the number of bodies is large.

One approach to simplify the problem is to assume the array to be made up of an infinite number of identical bodies which are equally spaced. Depending on the problem it may be useful to consider the array to extend infinitely in both directions or in one direction only. The first case, referred to as scattering by an infinite array, is applicable to wave scattering in the marginal ice zone. It has recently been investigated for bodies of arbitrary shape by Peter *et al.* (2006). The second case, referred to as scattering by a semi-infinite array, is of particular interest if edge effects are important but it is also the more complicated case. However, as suggested by Linton & Martin (2004), the utilisation of the solution of the scattering of the infinite array can very much simplify the (numerical) solution of the semi-infinite scattering in a great number of situations. The scattering by an infinite array can therefore be understood as a problem in its own right as well as a first step towards the solution of the scattering by a semi-infinite array.

We present here both the infinite and semi-infinite array cases in a unified manner. We look for the scattered wavefields of the bodies in the array in terms of the coefficients in the expansion in cylindrical eigenfunctions. Along the lines of general interaction theories (cf. Kagemoto & Yue, 1986; Peter & Meylan, 2004), a system of equations is derived for the unknown coefficients in the expansion. For the infinite array, due to the periodicity of the geometry of the setting as well as the ambient incident wave, it suffices to consider the scattered wavefield of one single body. The total scattered wavefield can be obtained from the solution of this single body. For the semi-infinite array, the solution of the infinite-array problem can be used to simplify the system of equations for the semi-infinite array. We are also interested in the far field describing the scattering far away from the array. For the infinite-array problem, there is a finite number of plane scattered waves travelling away from the array while for the semi-infinite problem, the same plane waves occur but only for some values of the observation angle and there is also a circular wave from the edge. This is also illustrated by simulation results.

## 2 Problem formulation

We consider the water-wave scattering of a plane wave by an infinite array or a semi-infinite array of identical vertically non-overlapping bodies, denoted by $\Delta_j$. The mean-centre positions $O_j$ of $\Delta_j$ are assumed to be $O_j = (jR, 0)$ where the distance between the (centres of the) bodies, $R$, is supposed sufficiently large so that there is no intersection of the smallest cylinder which contains each body with any other body. In the infinite-array case we have $j \in \mathbb{Z}$, for the semi-infinite array we assume $j \in \mathbb{N}$ (note that we have $0 \in \mathbb{N}$). The plane wave is assumed to travel in the direction $\chi \in (0, \pi/2]$ where $\chi$ is measured with respect to the $x$-axis. Let $(r_j, \theta_j, z)$ be the local cylindrical coordinates of the $j$th body, $\Delta_j$. Note that the zeroth body is centred at the origin and its local cylindrical coordinates coincide with the global ones $(r, \theta, z)$.

The equations of motion for the water are derived from the linearised inviscid theory. Only fixed radian frequencies $\omega$ are considered so the time-dependence of the water velocity potential is factored out, $\Phi(\mathbf{x}, t) = \mathbb{R}e\,\{\phi(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$. The undisturbed water surface is assumed at $z = 0$.

Writing $\alpha = \omega^2/g$ where $g$ is the acceleration due to gravity, for water of constant finite depth $d$, the potential $\phi$ has to satisfy the standard boundary-value problem,

$$\begin{aligned} \nabla^2\phi &= 0, && \mathbf{x} \in D, \\ \partial_z\phi &= \alpha\phi, && \mathbf{x} \in \Gamma^{\mathrm{f}}, \\ \partial_z\phi &= 0, && \mathbf{x} \in D,\ z = -d, \end{aligned}$$

where $D$ is the domain occupied by the water and $\Gamma^{\mathrm{f}}$ is the free water surface. At the immersed body surface $\Gamma_j$ of $\Delta_j$, the water velocity potential has to equal the normal velocity of the body $\mathbf{v}_j$,

$$\partial_n\phi = \mathbf{v}_j, \qquad \mathbf{x} \in \Gamma_j.$$

The system of equations is completed by appropriate radiation conditions.

For future reference, we note that the positive wavenumber $k$ is related to $\alpha$ by the dispersion relation

$$\alpha = k \tanh kd,$$

and the values of $k_m$, $m > 0$, are given as positive real roots of the dispersion relation

$$\alpha + k_m \tan k_m d = 0.$$

For ease of notation, we write $k_0 = -\mathrm{i}k$. Moreover, we denote the ambient incident potential by $\phi^{\mathrm{In}}$.

### 2.1 Eigenfunction expansion of the potential

In water of constant finite depth $d$, the scattered potential $\phi_j^{\mathrm{S}}(r_j, \theta_j, z)$ of a body $\Delta_j$ can be expanded in cylindrical eigenfunctions,

$$\phi_j^{\mathrm{S}} = \sum_{m=0}^{\infty} f_m(z) \sum_{\mu=-\infty}^{\infty} A_{m\mu}^{j} K_\mu(k_m r_j)\mathrm{e}^{\mathrm{i}\mu\theta_j}, \qquad (1)$$

with discrete coefficients $A_{m\mu}^j$ where $f_m(z) = (\cos k_m(z+d))/(\cos k_m d)$. The incident potential $\phi_j^{\mathrm{I}}(r_j, \theta_j, z)$ upon body $\Delta_j$ can also be expanded in cylindrical eigenfunctions,

$$\phi_j^{\mathrm{I}} = \sum_{n=0}^{\infty} f_m(z) \sum_{\nu=-\infty}^{\infty} D_{n\nu}^{j} I_\nu(k_n r_j)\mathrm{e}^{\mathrm{i}\nu\theta_j}, \qquad (2)$$

with discrete coefficients $D_{n\nu}^j$. The functions $I_\nu$ and $K_\nu$ denote the modified Bessel functions of the first and second kind, respectively.

## 3 The system of equations

Following the ideas of general interaction theories (Kagemoto & Yue, 1986; Peter & Meylan, 2004), a system of equations for the unknown coefficients of the scattered wavefields of all bodies is developed. This system of equations is based on transforming the scattered potential of $\Delta_j$ into an incident potential upon $\Delta_l$ ($j \neq l$). Doing this for all bodies simultaneously, and relating the incident and scattered potential for each body, a system of equations for the unknown coefficients is developed. This system of equations can then be simplified making use of the particular setting (i.e. infinite or semi-infinite array).

After some calculations (the details of which can be found in Kagemoto & Yue, 1986, e.g.) the following system of equations is obtained for the unknown coefficients of the scattered wavefields,

$$A_{m\mu}^{l} = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} B_{mn\mu\nu} \Big[ D_{ln\nu}^{\mathrm{In}} + \\ + \sum_{j \in \mathcal{X}_l} \sum_{\tau=-\infty}^{\infty} A_{n\tau}^{j} (-1)^\nu K_{\tau-\nu}(k_n |j-l| R) \mathrm{e}^{\mathrm{i}(\tau-\nu)\varphi_{j-l}} \Big], \qquad (3)$$

$m \in \mathbb{N}$, $\mu \in \mathbb{Z}$ and $l \in \mathcal{X}$ where $\mathcal{X} = \mathbb{Z}$ in the infinite-array case and $\mathcal{X} = \mathbb{N}$ for the semi-infinite array and $\mathcal{X}_l = \mathcal{X}\backslash\{l\}$. Here, $B$ is the diffraction transfer operator of the body relating the coefficients of the incident and scattered partial waves, $D_{ln\nu}^{\mathrm{In}}$ are the coefficients of the ambient incident wave in the expansion (2) and the angles $\varphi_n$ account for the difference in direction determining whether the $j$th body is located to the left or to the right of the $l$th body and are defined by $\varphi_n = 0$ for $n < 0$ and $\varphi_n = \pi$ for $n > 0$.

### 3.1 The infinite array

For the infinite array, we can use the periodicity of the geometry and of the incident wave to write the coefficients $A^l_{m\mu}$ as $A^l_{m\mu} = P_l A^0_{m\mu} = P_l A_{m\mu}$, say, where the phase factor $P_l$ is defined by

$$P_l = \mathrm{e}^{\mathrm{i}klR\cos\chi}.$$

The same can be done for the coefficients of the incident ambient wave, i.e. $D^{\mathrm{In}}_{ln\nu} = P_l D^{\mathrm{In}}_{n\nu}$. Introducing the constants

$$\sigma^n_\nu = \sum_{j=1}^{\infty} (P_{-j} + (-1)^\nu P_j) K_\nu(k_n jR),$$

which can be evaluated separately since they do not contain any unknowns, (3) reduces to

$$A_{m\mu} = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} B_{mn\mu\nu} \Big[ D^{\mathrm{In}}_{n\nu} + (-1)^\nu \sum_{\tau=-\infty}^{\infty} A_{n\tau}\, \sigma^n_{\tau-\nu} \Big]. \tag{4}$$

The efficient computation of the constants $\sigma^0_\nu$ is not trivial but appropriate methods are outlined in Peter *et al.* (2006).

### 3.2 The semi-infinite array

In order to avoid confusion, in what follows, we denote the coefficients of the bodies in the semi-infinite array by $X^j_{m\nu}$, $j \in \mathbb{N}$, and the difference between those of the infinite array and those of the semi-infinite array by

$$Z^j_{m\nu} = X^j_{m\nu} - P_j A_{m\nu}. \tag{5}$$

As $j$ becomes large, it can be expected that the coefficients of the infinite and the semi-infinite array become more and more similar implying that the $Z^j_{m\nu}$ get small. Instead of solving for the $X^j_{m\nu}$ directly, we derive a system of equations for the differences $Z^j_{m\nu}$ and solve for these. This allows us to only use a finite amount of bodies ($j = 0, \dots, N$) when (numerically) solving for the first $M$ bodies (where $M < N$).

It is important to note that the $Z^j_{m\nu}$ may not become small for increasing $j$ in the case that a Rayleigh-Bloch wave is excited which may happen if $k\cos\chi < \pi/R$ (Porter & Evans, 1999). It is a topic of current research to find conditions when Rayleigh-Bloch are excited in arrays of bodies of arbitrary shape and how to proceed if such a wave travels down the array. In our considerations, we assume that no Rayleigh-Bloch wave is excited. In any case, the convergence of the method can easily be checked by making sure that the $Z^j_{m\nu}$ become small for increasing $j$.

Substituting (5) into (3) with $\mathcal{X} = \mathbb{N}$ and using (4) we obtain

$$Z^l_{m\mu} = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{\infty} B_{mn\mu\nu} \Big[ \sum_{\substack{j=0\\ j\neq l}}^{\infty} \sum_{\vartheta=-\infty}^{\infty} Z^j_{n\vartheta} (-1)^\nu K_{\vartheta-\nu}(k_n|j-l|R) \mathrm{e}^{\mathrm{i}(\vartheta-\nu)\varphi_{j-l}} - (-1)^\nu P_l \sum_{\vartheta=-\infty}^{\infty} A_{n\vartheta} \tilde{\sigma}^{nl}_{\vartheta-\nu} \Big]$$

as the final system of equations where the constants $\tilde{\sigma}^{nl}_\nu$ are defined by

$$\tilde{\sigma}^{nl}_\nu = \sum_{j=l+1}^{\infty} P_{-j} K_\nu(k_n jR).$$

A method for the efficient computation of these constants when $n = 0$ has only recently been developed by one of the authors; see Linton (2005).

## 4 The far field

In this section, the far field is examined which describes the scattering far away from the array. First, we define the scattering angles which give the directions of propagation of plane scattered waves far away from both types of arrays.

Letting $p = 2\pi/R$, define the scattering angles $\chi_m$ by

$$\chi_m = \arccos(\psi_m/k) \text{ where } \psi_m = k\cos\chi + mp$$

and write $\psi$ for $\psi_0$. Also note that $\chi_0 = \chi$ by definition. If $|\psi_m| < k$, i.e. if

$$-1 < \cos\chi + \frac{mp}{k} < 1,$$

we say that $m \in \mathcal{M}$ and then $0 < \chi_m < \pi$. It turns out (see below) that these angles ($\pm\chi_m$ for $m \in \mathcal{M}$) are the directions in which plane waves propagate away from the array.

For a derivation of the far field of the infinite array, we refer to Peter *et al.* (2006). As $kr \to \infty$ away from the array axis $y = 0$, the far field consists of a set of plane waves propagating in the directions $\theta = \pm\chi_j$,

$$\phi \sim \phi^{\mathrm{In}} + \frac{\pi \mathrm{i}}{kR} \frac{\cosh k(z+d)}{\cosh kd} \sum_{j\in\mathcal{M}} \frac{1}{\sin\chi_j} \mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\chi_j)} \sum_{\mu=-\infty}^{\infty} A_{0\mu}\, \mathrm{e}^{\pm\mathrm{i}\mu\chi_j}.$$

For the semi-infinite problem, the same plane waves occur but only for some values of the observation angle and there is also a circular wave from the edge.

## 5 Results

In this section we present some calculations for arrays of ice floes. The calculation of the diffraction transfer matrices of ice floes is discussed in Peter & Meylan (2004).

We consider ice floes with non-dimensionalised stiffness $\beta = 0.2$ and mass $\gamma = 0.2$ (using the non-dimensionalisation of Peter & Meylan) in water of depth 1/2. The spacing is $R = 3$, the wavelength of the ambient incident wave is 2 and the side length of each square ice floe is one wavelength. The ambient wavefield is of unit amplitude and propagates in a direction making an angle of $\chi = \pi/5$ with the $x$-axis. Note that we have $k \cos \chi > \pi/R$ in this example.

Figure 1 shows the solution for the infinite array while figure 2 shows the solution for the semi-infinite array under the same conditions. In the plots of the scattered wavefields, the very special behaviour at the edge of the semi-infinite array can be observed. In particular, the additional circular wave can be seen clearly.

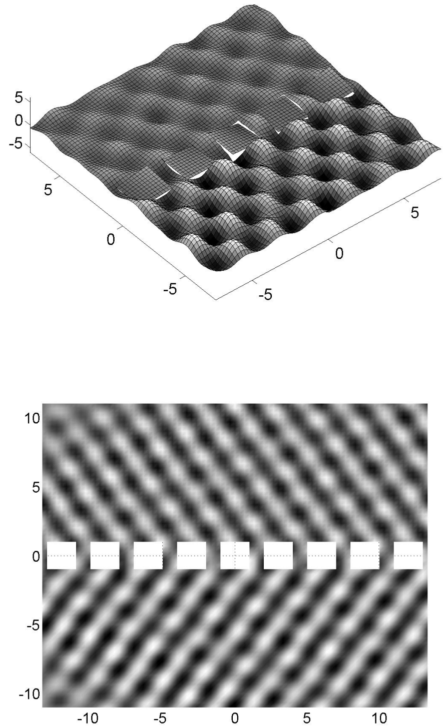


Fig. 1: Infinite array of ice floes (top) and scattered wavefield (bottom).

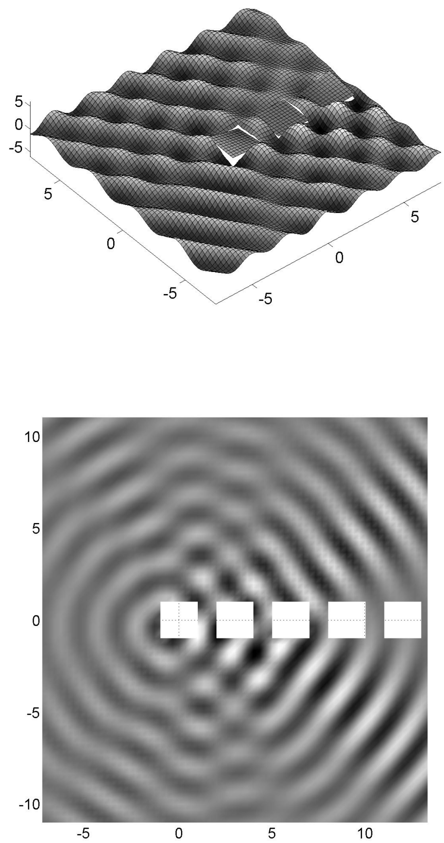


Fig. 2: Semi-infinite array of ice floes (top) and scattered wavefield (bottom).

## References

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# Peter, M.A., Meylam, M.H., Linton, C.M.
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## Discusser - D.V. Evans:

According to Wilcox in his book on Diffraction Gratings, Rayleigh Bloch waves do not exist if a Dirichlet condition is applied to the grating, but may exist if a Neumann condition is applied, although no absolute conditions are known.

**Reply:**

That is a very important point. We are aware of this fact and have also found that the body geometry seems to be of importance.

## Discusser - M. Kashiwagi:

For a large number of bodies, the hierarchical wave interaction theory developed by myself can be applied without using the periodicity assumption. In a real situation, the number of arrays must be finite. Why don't you consider to use the hierarchical wave interaction theory for the present problem?

**Reply:**

The hierarchical interaction theory is a powerful method for solving for a large number of bodies. It would apply well to the finite-array problem. However, it is numerically more complicated than the computations required to solve for the infinite and semi-infinite array the way we presented. Moreover, our computations - although only approximating the finite-array problem - yield quick good results with little numerical effort. If very accurate results are required for the finite array, the hierarchical interaction theory should be used.